Three lectures on low-dimensional phase transitions

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1 Why are lower dimensions interesting?

1.1 Exact solutions in low-dimensional systems

"Es wird gezeigt, dass ein solches Modell noch keine ferromagnetische Eigenschaften besitzt, und dieser Aussage auch auf das dreidimensionale Modell ausgedehnt." E. Ising, 1925

Two early, frequently quoted, papers on one-dimensional systems are Ising's 1925 paper on a chain of two-state spins in one dimension [1], and Bethe's 1931 paper on a chain of quantum spin 1/2 fermions [2]. The Ising argument was simple, and the Bethe argument subtle, but the solution by Onsager of the two-dimensional Ising model [3] combined features of both.

The 1D Ising model can be solved because the partition function $\mathcal{Z}$, the trace of $\exp(-\beta\mathcal{H})$, can be written as the trace of the product of the transfer matrix

$$\begin{pmatrix} \exp(\beta B) & \exp(\beta J) \\ \exp(\beta J) & \exp(-\beta B) \end{pmatrix},$$

for each pair of neighboring sites, where B is the strength of the magnetic field and J is the coupling between nearest neighbor sites. This trace is dominated by the largest eigenvalue of the transfer matrix.

For the two-dimensional Ising model the same sort of formalism has been used, but it is much more complicated, because the transfer matrix has to be written in terms of the relation between the states of successive columns of the spins. The exact solution of this was a major struggle. According to a talk by E. Montroll that I once heard, Onsager steadily increased the number of columns he included until he could guess the form of the answer and check that it was correct. This left other people in the field constructing simpler arguments, a few of which are compact enough to include in text-books.

There is a similar additional level of complication involved in Bethe's exploration of the eigenstates of the one-dimensional chain of Heisenberg spins. The method which was used to solve this one-dimensional quantum problem is known as the Bethe ansatz, and this method has been heavily quoted since 1970. I found over 2800 papers that have used this phrase since then.

In 1936 Peierls produced a proof, faulty in detail but using a powerful method, that the two- and three-dimensional Ising models are magnetized at low temperatures [4], and so we know that Ising's statement was false.

1.2 What else is known about critical phenomena?

Claims of exact solutions of the three-dimensional Ising model have emerged occasionally, but none has survived close scrutiny. The model can be studied experimentally by finding uniaxial magnetic systems, or by analysis of high temperature power series for the properties of an ideal Ising model.

Sceptics thought that low-dimensional systems were uninteresting, because real materials live in a three-dimensional world. Singular behavior of the specific heat and related quantities was a feature of many early experimental results, dating back to the 1870s, which had often been ignored, or, worse still, distorted to fit the results of the beautiful van der Waals theory. There was also an apparently careful analysis of bulk liquid helium close to the superfluid transition, which was incorrectly fitted to the logarithmic singularity of the two-dimensional Ising model by FBK.

From 1960 onwards experimentalists, using careful analysis of measurements, and theorists, analysing high temperature and low field expansions of simple models, turned to a serious empirical study of the neighborhoods of critical points. It emerged that there were a limited number of universality classes, each with characteristic values of the critical exponents. Gas-liquid critical points, which were first studied by Thomas Andrews, share a universality class with the three-dimensional Ising model, and with the ordering transition of β brass, a 50% mixture of copper and zinc atoms. The exponents for the two-dimensional Ising model have been found for adsorbates on suitable substrates, such as graphite, and experimental results are close to calculated values.

A common method of fitting a high temperature power series is to use Pade approximants, which are a succession of ratios of two power series designed to fit the original power series or its logarithm. This can lead to an accurate determination of the leading singularity of the free energy at the critical point of the phase transition.

A revolutionary approach to such problems was introduced in 1972 by Kenneth Wilson and Michael Fisher at Cornell University. The title of the paper is "Critical exponents in 3.99 dimensions" [5]. For many systems, 4 dimensions is where mean field gaussian behavior sets in, and the title expressed a hope that a perturbation away from 4 dimensions can lead to an understanding of what happens in 3 dimensions, or maybe even in 2 dimensions.

This led to an explosion of work on critical phenomena and on related phenomena in quantum field theory.

Mostly it worked well, but there were special difficulties with the three-dimensional Ising model. It turned out that at three dimensions there were confluent singularities rather than simple poles, so extrapolation required much more care. Bernie Nickel [6] and John Rehr [7] were two of the people who sorted this out.

It was also not easy to extrapolate away from known two dimensional solutions, but this provided an additional method of studying the critical behavior of three-dimensional systems.

1.3 Real systems with reduced dimensionality

A sample with rectangular faces behaves two-dimensionally when the temperature is such that the correlation length is long compared with the thickness, but short compared with the width and breadth.

A long sample with small regular cross-section can be one-dimensional if correlations are strong across the cross-section, but die off along the length.

There are various ways of approaching one or two-dimensional systems without reducing the number of atoms so much that the signals an experimenter needs to measure are too small to measure accurately.

Many layers with varying width of intercalating layers, eg. in graphite. Mildred Dresselhaus [8] has been one of the leaders in the study of intercalated graphite, and was recently awarded the Kavli Prize for her work on graphite.

Multiple layers of graphene with varying separation, are used to approach the independent layer limit, while maintaing a strong total signal.

Similar things can be done with multiple magnetic layers of varying separation. There was a good review in 1974 by de Jongh and Miedema [9].

There can be complicated interconnections across sample, as for helium films in vycor glass [10] or in gels. Does the complicated interconnectivity make it a bulk sample with holes in it? Studies of superfluid helium in such environments by Moses Chan show that the behavior in such multiply connected thin films is like that of bulk materials, but the superfluid transition occurs at a much lower critical temperature.

Cold atoms allow large increase in signal strength per atom, by using resonant samples with narrow line-width.

1.4 Dimensionality in superconductors

Superconductors have a lot of interconnected coherence lengths involved. Mean free path of electrons, size of electron pair, penetration depth of magnetic field, etc.

In 1936 it was discovered by Shubnikov and colleagues [12, 13] that superconductors could be divided into two types. The Type I materials were mostly clean materials in which the magnetic field was expelled from the superconducting regions, while Type II materials were mostly alloys (apart from pure niobium and vanadium), and the magnetic field coexisted with superconductivity.

Shubnikov was killed, and the community had more urgent problems to think about for the next ten years, until the theory was rediscovered by Pippard, Gorkov, and others.

Superconducting magnets have to be made from Type II materials. There is phase separation between the magnetic regions and the superconducting regions.

Cuprate superconductors are strongly type II.

1.5 Electronic devices and memory units

Most electronic devices made in the past fifty years are based on transistors in which the electrons are trapped on a plane interface. Good GaAs device has long electron mean free path, which is why your cell phone can pick up a tiny signal.

The giant magnetoresistance units used for memory devices in the past twenty years or so also need plane geometry for easy read-out.

2 Theory of BKT transitions: 1. Early work

2.1 Line defects and phase transitions

In Onsager's 1949 31-line "Observation" [14] he states, among other things, that circulation in superfluids is quantized in multiples of h/m_A, where m_A is the atomic mass, and that the circulation can either be round a solid obstacle or round a line singularity within the superfluid. He also says "As a possible interpretation of the λ-point, we can understand that when the concentration of vortices reaches the point where they form a connected tangle throughout the liquid, then the liquid becomes normal." It should be no surprise that Onsager's results were not widely appreciated until Feynman rederived them four or five years later [15].

A similar suggestion has been made about the melting of solids, that a tangle of dislocations might form spontaneously at the melting temperature. This theory is discussed in Nabarro's *Theory of Crystal Dislocations* (Oxford, 1967), pages 688-691.

As far as I know, there is no compelling model based on defect driven phase transitions in three dimensions. The statistical mechanics of line defects is complicated, and this is an obstacle to making a detailed theory.

2.2 One-dimensional Ising model with long-range interactions

What happens for the one-dimensional Ising model with a $1/r^2$ interaction between spins? This question was posed to me by Phil Anderson, whose work on the Kondo effect depended on this [16, 17]. It was known, from work by Ruelle and by Dyson [18, 19], that there is no phase transition if the interaction falls off faster than $1/r^2$ and there is a magnetized state if the fall-off is significantly slower than this.

On the last page of Landau–Lifshitz *Statistical Physics* [20] there is an argument for the impossibility of having different phases in one-dimensional systems, and this argument can be modified for the case of long range interactions. In a system of length L with $2p$ domain walls, the entropy is

$$S \approx k_B p \ln(L/a)\,, \tag{1}$$

where a is the spacing between spins. In the particular case of a Ja^2/r^2 interaction between spins, the energy of a pair of domain walls at x_1, x_2 can be approximated as

$$E = 2J\mu^2 \int_{-L/2}^{x-a/2} \int_{x+a/2}^{L/2} (x_2 - x_1)^{-2} dx_2 dx_1 \approx 2J\mu^2 \ln(L/4a)\,, \tag{2}$$

where μ is the magnetization per spin of the system. At low temperatures the free energy $F = E - TS$ is dominated for large L by the energy, so there are no free domain walls in equilibrium, but for $T > J/k_B$ the entropy dominates, free domain walls are thermodynamically stable, and there is no net magnetization.

This implies that the magnetization cannot go steadily to zero as the temperature is increased, but jumps to zero when its minimum allowed value is reached [21]. This shaky argument was improved by Dyson in 1971 [22] and made rigorous by Fröhlich and Spencer [23]. There were signs of such a discontinuity in the equilibrium magnetization in simulations by Rapaport and Frankel in Melbourne [24], and in related work by Nagle and Bonner [25] in Pittsburgh.

2.3 Superfluid films

Onsager's interpretation of the two-fluid model of superfluid is that there is a condensate wave function of constant magnitude, whose phase S satisfies the Laplace equation, and whose velocity is given by

$$\mathbf{v}_s = (\hbar/M_A)\mathbf{grad}S\,; \tag{3}$$

this superfluid component also carries a normal fluid of thermal excitations, such as phonons and rotons. The condensate may also contain vortex lines, which are singularities of the condensate wave function, around which the phase changes by a multiple of 2π. This leads to quantized circulation of the superfluid flow in multiples of h/M_A. The only degrees of freedom of a quantized vortex are the two coordinates of the position of the vortex core, which, in a classical theory of vortex dynamics, are conjugate dynamical variables, and which should therefore be governed by an uncertainty relation in a quantized theory. This uncertainty relation quantizes the total area πR^2 into cells of area $\pi a_1^2 = M_A/\rho_s$, and this gives the entropy per vortex as

$$S_v \approx k_B \ln(L^2/a_1^2)\,. \tag{4}$$

The vortices can have positive or negative circulation $\pm h/M_A$. The energy of an isolated vortex near the center of a disk of superfluid of radius R can be written as

$$E_V \approx 2\pi \int_{a_0}^{R} \rho_s \Big(\frac{\hbar}{M_A r}\Big)^2 r dr = 2\pi \rho_s (\hbar/M_A)^2 \ln(R/a_0) \,. \qquad (5)$$

Here a_0 is the vortex core radius, the healing distance over which the superfluid density is suppressed by the high speed near the core, and ρ_s is the superfluid density per unit area, which is *defined* and measured in terms of the energy density for a given superfluid flow velocity.

There is obviously a close analogy between Eqs. (1), (2) for the one-dimensional Ising model with $1/r^2$ interactions, and Eqs. (4), (5), for vortices in a superfluid thin film. In analogy with Eq. (2) for the Ising model, the energy of a pair of oppositely rotating vortices with circulation $\pm h/M_A$ separated by a distance d is given as

$$E_{+-}(d) = 2\pi\rho_s(\frac{\hbar}{M_A})^2 \ln\frac{d}{a_0} \,. \qquad (6)$$

Comparison of Eqs. (4) and (5) shows that the condition for stability of the superfluid against the appearance of isolated vortices is

$$k_B T \leq \pi\rho_s(\hbar/M_A)^2 \,. \qquad (7)$$

These arguments were given in our 1972 and 1973 papers [26, 27]. However we got an incorrect factor of M_A/M^*, where M^* is an effective mass, in our derivation. A proper derivation was given by Nelson and Kosterlitz [28]. The corrected result is important, because the parameter-free relation between the critical temperature and superfluid density can be, and was, experimentally verified. A straightforward explanation for this robust relation between the maximum temperature for stable superfluidity comes from the fact that the coefficient of the logarithm in the expression (5) depends on the average flow induced on a circuit at large distance from the vortex core by the 2π twist of the phase angle, and the superfluid density per unit area is also defined by the energy induced by a twist of the phase angle imposed over a large area. Therefore, even if nonuniformity of the substrate causes nonuniformity of the superfluid film, it is the *same* average that comes into the expression for superfluid density and for vortex energy.

It was nice to think that we knew something about helium films that no-one else had thought of, but on a visit to Paris I was told by Paul Martin, from Harvard, that a Russian visitor had told him of similar work by Vadim Berezinskii, done about a year earlier than our work.

There are close analogies between the vortex pairs in superfluid films, and the domain wall pairs in the long-range one-dimensional Ising model. In the low-temperature phase vortex pairs and domain wall pairs remain bound. In the high temperature phase there are unpaired vortices and unpaired domain walls that can move to destroy long range superfluid order or magnetic order. There is one important difference which is that we know no physical system that behaves like the long-range Ising model, but we know that low temperature liquid helium films are superfluid.

I tried to persuade an experimentalist friend to measure this, but he told me that the measurements had already been made, the discontinuity had been seen, but more careful analysis had made the discontinuity in superfluid density disappear. The moral I would draw from this story is that experimenters should not seek too carefully for agreement with existing theory, nor should theorists seek too carefully for agreement with known experimental results. I was pleased when I heard Joe Vinen rebuke a member of his group who quoted agreement with theory as a sign of the accuracy of his measurements.

A rigorous argument for the discontinuity of superfluid density was given by Fröhlich and Spencer [32], just before their paper about the one-dimensional magnet.

If we regard the isolated vortex, or the isolated domain wall, as a monopole, and bound pairs as dipoles, the low temperature phase, in both the one-dimensional and two-dimensional cases, is a phase with no free monopoles, while the high temperature phase has an equilibrium distribution of free monopoles. As the temperature approaches the critical temperature for free monopoles from below, the typical size of dipoles increases, and diverges at the critical temperature. A renormalization group theory of this transition, modeled on the theories developed for more familiar critical points, was developed by Kosterlitz in 1974.

In the low temperature phase of the superfluid the addition of potential flow to the system changes the equilibrium configuration of the vortices, and may cause flow of the vortices if there are not enough pinning centers for the vortices, but it does not cause dissipative transfer of energy to the phonon–roton system. Above the critical temperature energy is transferred to the phonon–roton system by moving positive vortices relative to negative vortices.

BISHOP & REPPY (1978)

SUPERFLUID DENSITY JUMP AS A FUNCTION OF TRANSITION TEMPERATURE

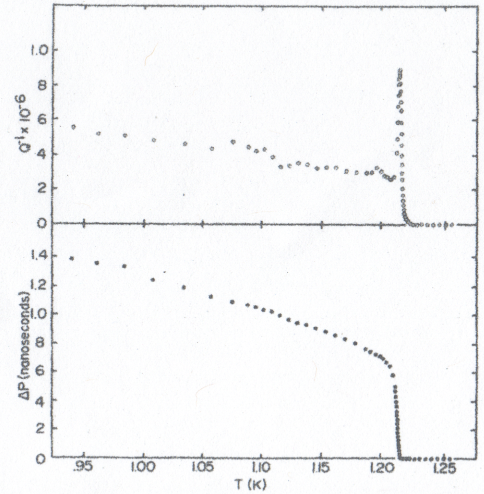

FIG. 1. The shift in period, ΔP, and dissipation Q^{-1} are shown as a function of temperature at the superfluid transition.

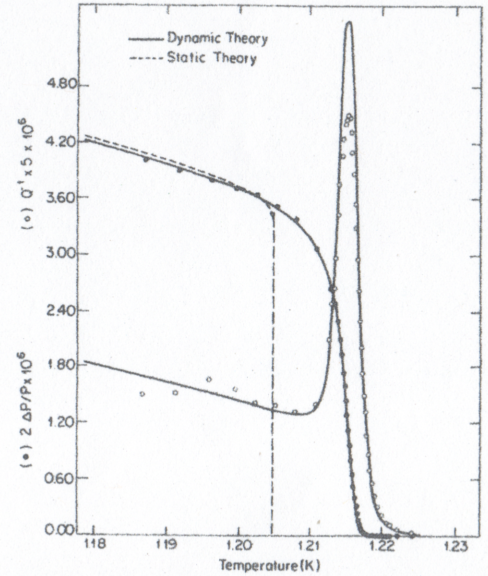

FIG. 2. The reduced period shift, $2\Delta P/P$, and dissipation Q^{-1} are shown for a superfluid transition temperature of 1.215 K. The solid lines are fits using the dynamic theory of AHNS (Ref. 6) and the dashed curve is the result of the static theory.

rcor glass,[1] exhibit any excess dissipa-
:iated with the superfluid transition.
in dissipation in the present experiment

SHIFT IN OSCILLATOR PERIOD AND DAMPING AT SUPERFLUID → NORMAL FLUID TRANSITION

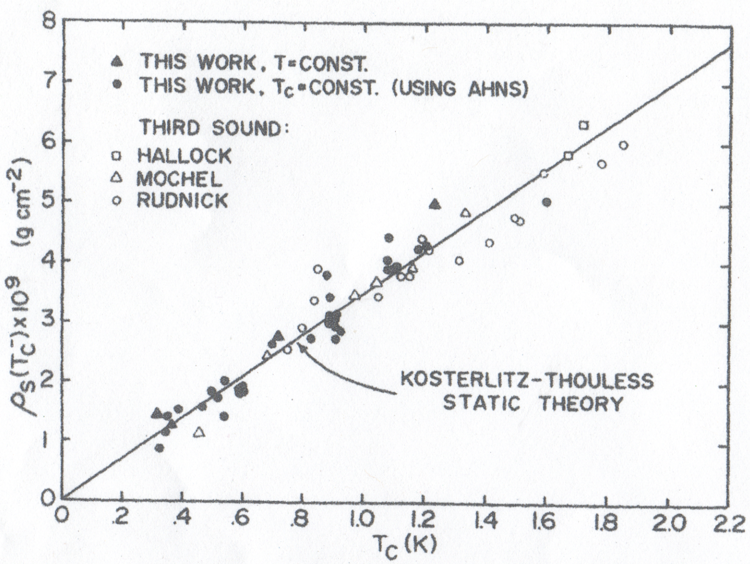

FIG. 3. Results of all of our data, in addition to previous third-sound results for the discontinuous superfluid density jump $\rho_s(T_c^-)$ as a function of temperature. The solid line is the Kosterlitz-Thouless (Refs.

BOSE CONDENSATE IS IRROTATIONAL

2.4 Experimental tests of the theory

The critical temperature at which some vortex pairs break up into two free vortices was predicted to be given by

$$k_B T_c = \pi \rho_s (\hbar / M_A)^2 \tag{8}$$

which is an equation with no free parameters. Experimental confirmation or correction of this relation required measurements of the temperature dependence of the superfluid density of helium films. One basic method was suggested by Landau and developed by Andronikashvili, which involved a stack of coaxial discs mounted in a cylinder filled with helium. The cylinder was suspended on a torsion fiber, and the oscillation frequency and damping rate were measured as a function of temperature. Above the critical temperature for superfluid onset, viscosity ensures that the fluid moves with the cylinder and discs, but below the critical temperature the moment of inertia drops, because the superfluid component does not follow the rotational motion of the apparatus. This allows the superfluid density to be measured as a function of temperature.

The use of a nonzero frequency makes the measurement of moment of inertia much more sensitive, but broadens the transition between superfluid and normal fluid behavior: the nonzero frequency theory was derived by Ambegaokar et al. [33]. Bishop and Reppy [34] used a roll of Mylar film rather than coaxial discs to maximize the inertia of the helium film relative to the substrate. The response of the mylar roll to a periodic drive was measured under varying loads of condensed helium gas on the film. The Mylar film was about 6 μm thick and 21 m long, and was driven at 2.5 kHz. The nonzero drive frequency increases the sensitivity of measurement, but leads to a raising of the transition temperature and broadening of the peak in attenuation at the transition.

The results of this experiment are plotted together with the results from third sound measurements, and can be seen on the next page. In the upper right corner can be seen plots of the damping and frequency plotted against temperature, compared with the nonzero frequency predictions of ref. [33], which give a broadened peak in the damping, and an upward shift in the apparent transitions temperature. In the bottom left corner are shown a comparison of these results for the dependence of transition temperature on film thickness, both for this experiment and for three different third sound experiments, which shows good agreement with the parameter-free prediction.

The figure shows a single dislocation in a triangular lattice, which is characterized a pair of neighboring atoms, one of which has seven neighbors and the other five whereas most atoms have six neighbors. The dislocation is characterized by its Burgers vector, which is the gap between the ends of a neighbor to neighbor regular polygon drawn around the dislocation.

Edge dislocation made up of two adjacent disclinations

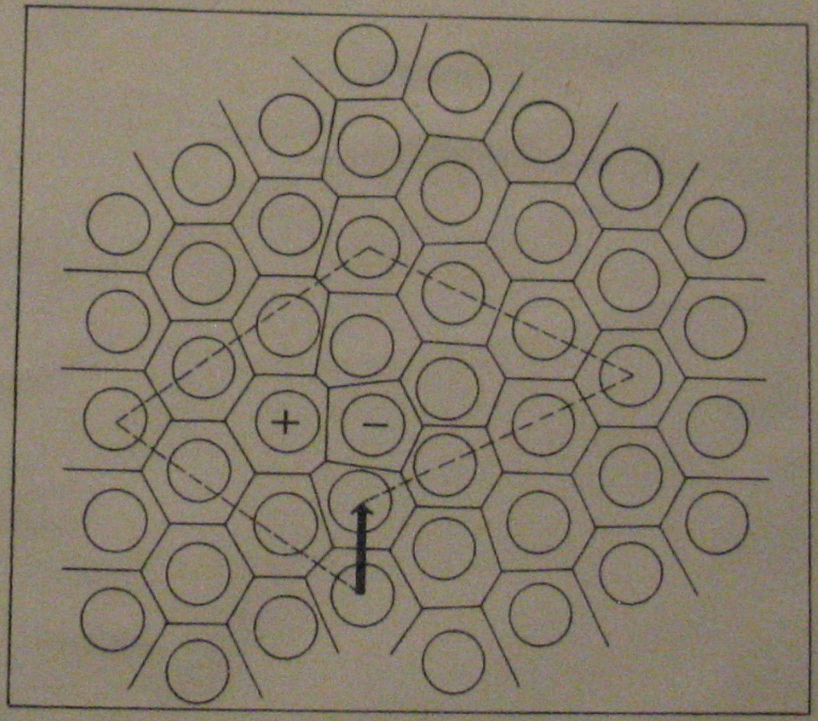

Fig. 1: The Dirichlet domain construction. A 7-5 pair is illustrated, with the path around the dislocation defining the Burgers vector.

McTague et al. 1980

3 BKT transitions: Later developments

We had written about some other possible applications of the theory, some of which were right, and some seriously wrong. The analogy of a vortex pair to a classical electric dipole which could either be bound or ionized was helpful to our thinking, as we had learned about such things when we were undergraduates, studying the theory of dielectrics and electrical conductors.

3.1 Planar magnets

The parallel case of a planar magnetic system with a preferred plane of polarization is important theoretically, since, in renormalization group analyses in the style of Wilson and Fisher, magnetic systems are often used in the definition of universality classes. However, for the bosonic fluid we do not usually expect breaking of U1 symmetry (uniform change of the phase of the wave function), but in real planar magnets there are crystal symmetries which break the rotational symmetry about the z-axis. There are therefore important studies of the relevance or irrelevance of such symmetry breaking terms in such magnetic systems.

3.2 Superconductors

Perversely we argued that a superconducting thin film should not display an abrupt transition, because the interaction between vortices is modified at large distances by screening of the magnetic field, which leads to a finite penetration depth for the field. We should have remembered that this penetration depth is very large in thin films, so that modification of the theory by the screening is unimportant. This was pointed out by Beasley, Moij and Orlando in 1979 [35]. Early experimental verification was published by Hebard and Fiory in 1983 [36].

3.3 Solids, hexatic liquid crystals and liquids

One of the most important errors that we made was to assume that the two-dimensional melting transition was equivalent to the superfluid transition, with edge dislocations taking the place of quantized vortices. This error was pointed out by Halperin and Nelson [37]. A concentration of free edge dislocations destroys the long range positional order of a solid, so that the Bragg peaks of X-ray scattering are smeared out.

The figure on the previous page shows a single dislocation in a triangular lattice, which is characterized a pair of neighboring atoms, one of which has seven neighbors and the other five, whereas most atoms have six neighbors. A single dislocation, or a low concentration of randomly oriented dislocations, does not change the overall orientational symmetry of the crystal. It does, however, destroy its translational symmetry, so that the Bragg peaks in X-ray scattering are no longer sharp. The dislocation is characterized by its Burgers vector, which can be found by trying to draw a regular hexagon from neighbor to neighbor around the dislocation. The Burgers vector is the gap between the two ends of this construction. For the dislocation shown in the diagram the Burgers vector is one nearest neighbor spacing in the vertical direction.

At low temperatures, dislocations are bound to one another in pairs, and the interaction between them has an angular dependence, unlike the isotropic interaction between vortices in superfluids; this is the true solid phase. Above the lowest critical temperature the dislocation pairs or triplets (for the case of three dislocations whose Burgers vectors add up to zero) dissociate to form a gas of free dislocations.

A nonzero concentration of dislocations destroys the positional order characteristic of a crystal (sharp Bragg peaks in the X-ray scattering pattern), but does not destroy the orientational order of the crystal. Halperin and Nelson called this state of matter with orientational order but no positional order a *hexatic liquid crystal*. There are materials in which such transitions occur.

A second, higher temperature transition may occur, in which pairs of *disclinations* (atoms with five or seven neighbors) dissociate to form a gas of disclinations. Such disclinations give local regions of seven- or five-fold symmetry, and so do destroy the overall crystal anisotropy, to produce a phase with the overall symmetry of a liquid.

3.4 Roughening of crystal facets

In 1976, Chui and Weeks found that the roughening of a facet of a three-dimensional crystal was a phase transition in the same class as the superfluid to normal transition in a helium film. This was a surprise, because the obvious representation of the facet free energy does not make this apparent. The two-dimensional Fourier components of the height of the interface from the ideal plane facet can be used to express the energy of a particular configuration of the interface, and this can be used to express the free energy as a gaussian integral, but a further change of variables must be made to bring it into the form which arises naturally in the other examples we have considered.

3.5 Final remarks

Michael Kosterlitz remained active in this area considerably longer than I did. Our last published joint work was a review in 1978 of two-dimensional physics published in *Progress in Low Temperature Physics*. Recently we contributed a chapter for a volume to be published by World Scientific Publishing Company, edited by Jorge Jose, on *Forty Years of BKT Theory*.

Acknowledgements

Firstly I want to acknowledge the teaching, guidance and friendship I have received over the past fifty years from Phil Anderson and Joe Vinen. Secondly, I owe a great deal to my friend and collaborator Mike Kosterlitz, who usually paid more attention to the details than I did. Thirdly, I want to thank the large number of people who took our ideas seriously, and turned our sometimes disjointed thoughts into real physics, particularly John Reppy and David Bishop, who did the right experiment at the right time, and those theorists who gave us great support in the early days, David Nelson, Michael Fisher, Jorge Jose, Leo Kadanoff, Scott Kirkpatrick, Bert Halperin, Vinay Ambegaokar, Eric Siggia, and many others.

I deeply regret never having had the opportunity to meet the physicist who got there before us, Vadim Berezinskii.

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