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Condensed Matter Journal Club
University of Florida at Gainesville

Non-equilibrium time evolution of bosons from the functional renormalization group

Peter Kopietz, Universität Frankfurt

Collaboration: T. Kloss , J. Hick (Frankfurt), Andreas Kreisel (now Gainesville)

SFB/Transregio 49
Frankfurt-Kaiserslautern-Mainz
Condensed matter systems with variable many-body interactions

outline:

1. Functional integral formulation of Keldysh technique
2. Non-equilibrium FRG
3. Magnons in Yttrium-Iron-Garnet, parametric resonance
4. FRG for toy model for lowest magnon mode in YIG
5. Thermalization of magnons in YIG

1. functional integral formulation of the Keldysh technique

(A. Kamenev, Les Houches, 2004; Hick, Kloss, PK, PRB 2013)

- model: magnons coupled to phonons:

$$\mathcal{H} = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b^\dagger_{\boldsymbol{q}} b_{\boldsymbol{q}} + \frac{1}{\sqrt{V}} \sum_{\boldsymbol{q}} U_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} (b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}})$$

magnon phonon $U_{\boldsymbol{q}} = U_0 \sqrt{\omega_{\boldsymbol{q}}}$ $\rho_{\boldsymbol{q}} = \sum_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}+\boldsymbol{q}}$

- non-equilibrium Green functions:

$$iG^R_{\boldsymbol{k}}(t,t') = \Theta(t-t')\langle [a_{\boldsymbol{k}}(t), a^\dagger_{\boldsymbol{k}}(t')]\rangle,$$
$$iG^A_{\boldsymbol{k}}(t,t') = -\Theta(t'-t)\langle [a_{\boldsymbol{k}}(t), a^\dagger_{\boldsymbol{k}}(t')]\rangle,$$
$$iG^K_{\boldsymbol{k}}(t,t') = \langle \{a_{\boldsymbol{k}}(t), a^\dagger_{\boldsymbol{k}}(t')\}\rangle,$$

$$iG^K_{\boldsymbol{k}}(t,t) = 1 + 2n_{\boldsymbol{k}}(t)$$

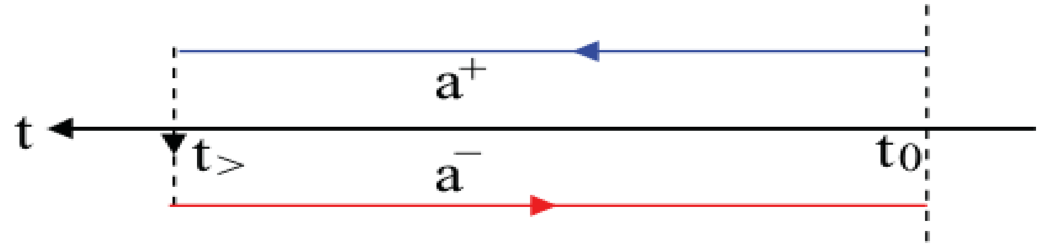
$$X_{\boldsymbol{q}} = \frac{1}{\sqrt{2}}\left(b_{\boldsymbol{q}} + b^\dagger_{-\boldsymbol{q}}\right)$$

$$iF^R_{\boldsymbol{q}}(t,t') = \Theta(t-t')\langle [b_{\boldsymbol{q}}(t), b^\dagger_{\boldsymbol{q}}(t')]\rangle,$$
$$iF^A_{\boldsymbol{k}}(t,t') = -\Theta(t'-t)\langle [b_{\boldsymbol{q}}(t), b^\dagger_{\boldsymbol{q}}(t')]\rangle,$$
$$iF^K_{\boldsymbol{k}}(t,t') = \langle \{b_{\boldsymbol{q}}(t), b^\dagger_{\boldsymbol{q}}(t')\}\rangle.$$

$$iD^R_{\boldsymbol{q}}(t,t') = \Theta(t-t')\langle [X_{\boldsymbol{q}}(t), X_{-\boldsymbol{q}}(t')]\rangle,$$
$$iD^A_{\boldsymbol{q}}(t,t') = -\Theta(t'-t)\langle [X_{\boldsymbol{q}}(t), X_{-\boldsymbol{q}}(t')]\rangle$$
$$iD^K_{\boldsymbol{q}}(t,t') = \langle \{X_{\boldsymbol{q}}(t), X_{-\boldsymbol{q}}(t')\}\rangle.$$

Keldysh contour and classical/quantum components

- Keldysh contour:



- change of basis: from contour labels to classical and quantum labels:

$$a^C_{\mathbf{k}}(t) = \frac{1}{\sqrt{2}}\left[a^+_{\mathbf{k}}(t) + a^-_{\mathbf{k}}(t)\right]$$

$$a^Q_{\mathbf{k}}(t) = \frac{1}{\sqrt{2}}\left[a^+_{\mathbf{k}}(t) - a^-_{\mathbf{k}}(t)\right]$$

- functional integral representation of non-equilibrium Green functions:

$$iG^R_{\mathbf{k}}(t,t') = \langle a^C_{\mathbf{k}}(t)\bar{a}^Q_{\mathbf{k}}(t')\rangle \equiv iG^{CQ}_{\mathbf{k}}(t,t')$$

$$iG^A_{\mathbf{k}}(t,t') = \langle a^Q_{\mathbf{k}}(t)\bar{a}^C_{\mathbf{k}}(t')\rangle \equiv iG^{QC}_{\mathbf{k}}(t,t')$$

$$iG^K_{\mathbf{k}}(t,t') = \langle a^C_{\mathbf{k}}(t)\bar{a}^C_{\mathbf{k}}(t')\rangle \equiv iG^{CC}_{\mathbf{k}}(t,t')$$

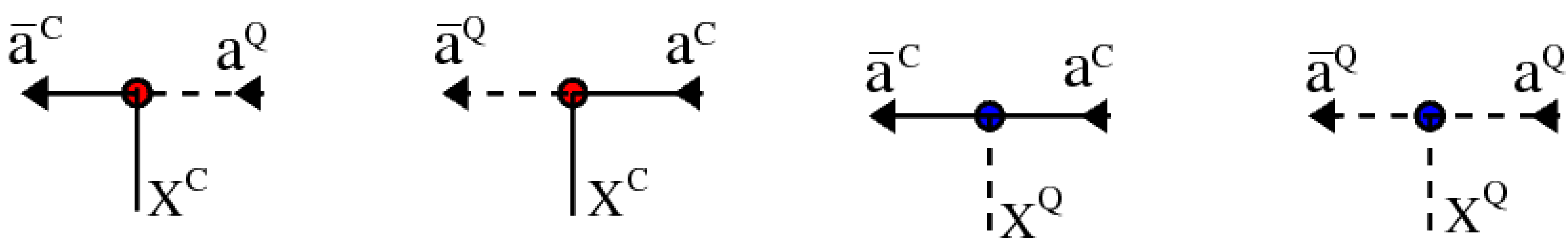
$$\langle a^\lambda_{\mathbf{k}}(t)\bar{a}^{\lambda'}_{\mathbf{k}}(t')\rangle = \int \mathcal{D}[a,\bar{a},b,\bar{b}]e^{iS[\bar{a},a,\bar{b},b]}a^\lambda_{\mathbf{k}}(t)\bar{a}^{\lambda'}_{\mathbf{k}}(t')$$

Keldysh action in continuum notation

$$S[\bar{a},a,\bar{b},b] = \int dt \int dt' \Bigg\{ \sum_{\boldsymbol{k}} (\bar{a}^C_{\boldsymbol{k}}(t), \bar{a}^Q_{\boldsymbol{k}}(t)) \begin{pmatrix} 0 & (\hat{G}^A_0)^{-1} \\ (\hat{G}^R_0)^{-1} & -(\hat{G}^R_0)^{-1}\hat{G}^K_0(\hat{G}^A_0)^{-1} \end{pmatrix}_{tt'} \begin{pmatrix} a^C_{\boldsymbol{k}}(t') \\ a^Q_{\boldsymbol{k}}(t') \end{pmatrix}$$

$$+ \sum_{\boldsymbol{q}} (\bar{b}^C_{\boldsymbol{q}}(t), \bar{b}^Q_{\boldsymbol{q}}(t)) \begin{pmatrix} 0 & (\hat{F}^A_0)^{-1} \\ (\hat{F}^R_0)^{-1} & -(\hat{F}^R_0)^{-1}\hat{F}^K_0(\hat{F}^A_0)^{-1} \end{pmatrix}_{tt'} \begin{pmatrix} b^C_{\boldsymbol{q}}(t') \\ b^Q_{\boldsymbol{q}}(t') \end{pmatrix} \Bigg\}$$

$$+ \int dt \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k},\boldsymbol{q}} U_{\boldsymbol{q}} \left[\left(\bar{a}^C_{\boldsymbol{k}+\boldsymbol{q}} a^Q_{\boldsymbol{k}} + \bar{a}^Q_{\boldsymbol{k}+\boldsymbol{q}} a^C_{\boldsymbol{k}} \right) X^C_{\boldsymbol{q}} + \left(\bar{a}^C_{\boldsymbol{k}+\boldsymbol{q}} a^C_{\boldsymbol{k}} + \bar{a}^Q_{\boldsymbol{k}+\boldsymbol{q}} a^Q_{\boldsymbol{k}} \right) X^Q_{\boldsymbol{q}} \right].$$



- QQ-blocks of Gaussian propagators are infinitesimal regularization:

$$-(\hat{G}^R_0)^{-1}\hat{G}^K_0(\hat{G}^A_0)^{-1} = 2i\eta\hat{g}_0$$

$$-(\hat{F}^R_0)^{-1}\hat{F}^K_0(\hat{F}^A_0)^{-1} = 2i\eta\hat{f}_0$$

$$[\hat{g}_0]_{tt'} = \delta(t-t')g_0 = \delta(t-t')[1+2\langle a^\dagger a\rangle_0]$$

$$[\hat{f}_0]_{tt'} = \delta(t-t')f_0 = \delta(t-t')[1+2\langle b^\dagger b\rangle_0]$$

non-equilibrium time-evolution: quantum kinetic equations

- Keldysh component of non-equilibrium Dyson equation gives kinetic equation for distribution function:

- Green function matrix: $\mathbf{G} = \begin{pmatrix} [\mathbf{G}]^{CC} & [\mathbf{G}]^{CQ} \\ [\mathbf{G}]^{QC} & 0 \end{pmatrix} = \begin{pmatrix} \hat{G}^K & \hat{G}^R \\ \hat{G}^A & 0 \end{pmatrix}$

matrices in momentum and time

- self-energy matrix: $\mathbf{\Sigma} = \begin{pmatrix} 0 & [\mathbf{\Sigma}]^{CQ} \\ [\mathbf{\Sigma}]^{QC} & [\mathbf{\Sigma}]^{QQ} \end{pmatrix} = \begin{pmatrix} 0 & \hat{\Sigma}^A \\ \hat{\Sigma}^R & \hat{\Sigma}^K \end{pmatrix}$

$$\hat{\Sigma}^K = -[\mathbf{G}^{-1}]^{QQ} = (\hat{G}^R)^{-1}\hat{G}^K(\hat{G}^A)^{-1}$$

- subtracting left/right Dyson eqns. gives kinetic eq.:

$$(\mathbf{G}_0^{-1} - \mathbf{\Sigma})\,\mathbf{G} = \mathbf{I} \qquad \left[\hat{M}_0, \hat{G}^K\right] = \hat{\Sigma}^K\hat{G}^A - \hat{G}^R\hat{\Sigma}^K + \hat{\Sigma}^R\hat{G}^K - \hat{G}^K\hat{\Sigma}^A$$

$$\mathbf{G}\,(\mathbf{G}_0^{-1} - \mathbf{\Sigma}) = \mathbf{I} \qquad [\hat{M}_0]_{\boldsymbol{k}t,\boldsymbol{k}'t'} = \delta_{\boldsymbol{k},\boldsymbol{k}'}\left[i\partial_t - \epsilon_{\boldsymbol{k}}\right]\delta(t-t')$$

different forms of the kinetic equation

1.) time-domain:

$$(i\partial_t + i\partial_{t'})G^K(t,t') = \int_{t_0}^{t} dt_1 [\Sigma^R(t,t_1)G^K(t_1,t') - G^R(t,t_1)\Sigma^K(t_1,t')] + \int_{t_0}^{t'} dt_1 [\Sigma^K(t,t_1)G^A(t_1,t') - G^K(t,t_1)\Sigma^A(t_1,t')]$$

2.) with subtractions to identify collision integrals:

$$\hat{\Sigma}^M = \frac{1}{2}[\hat{\Sigma}^R + \hat{\Sigma}^A] \qquad \hat{G}^M = \frac{1}{2}[\hat{G}^R + \hat{G}^A] \qquad \hat{M} = \hat{M}_0 - \hat{\Sigma}^M$$

$$\hat{\Sigma}^I = i[\hat{\Sigma}^R - \hat{\Sigma}^A] \qquad \hat{G}^I = i[\hat{G}^R - \hat{G}^A]$$

$$[\hat{M}, \hat{G}^K] - [\hat{\Sigma}^K, \hat{G}^M] = \hat{C}^{\text{in}} - \hat{C}^{\text{out}}$$

$$\hat{C}^{\text{in}} = \frac{i}{2}\{\hat{\Sigma}^K, \hat{G}^I\}$$

$$\hat{C}^{\text{out}} = \frac{i}{2}\{\hat{\Sigma}^I, \hat{G}^K\}$$

....different forms of the kinetic equation

3.) for distribution function: $\hat{G}^K = \hat{G}^R \hat{g}^\dagger - \hat{g}\hat{G}^A$

$$-i(\hat{M}\hat{g} - \hat{g}^\dagger \hat{M}) = \hat{\Sigma}^{\text{in}} - \hat{\Sigma}^{\text{out}}$$

$$\hat{\Sigma}^{\text{in}} = i\hat{\Sigma}^K$$

$$\hat{\Sigma}^{\text{out}} = \frac{1}{2}\left(\hat{\Sigma}^I \hat{g} + \hat{g}^\dagger \hat{\Sigma}^I\right)$$

4.) for distribution function, Wigner transformed:

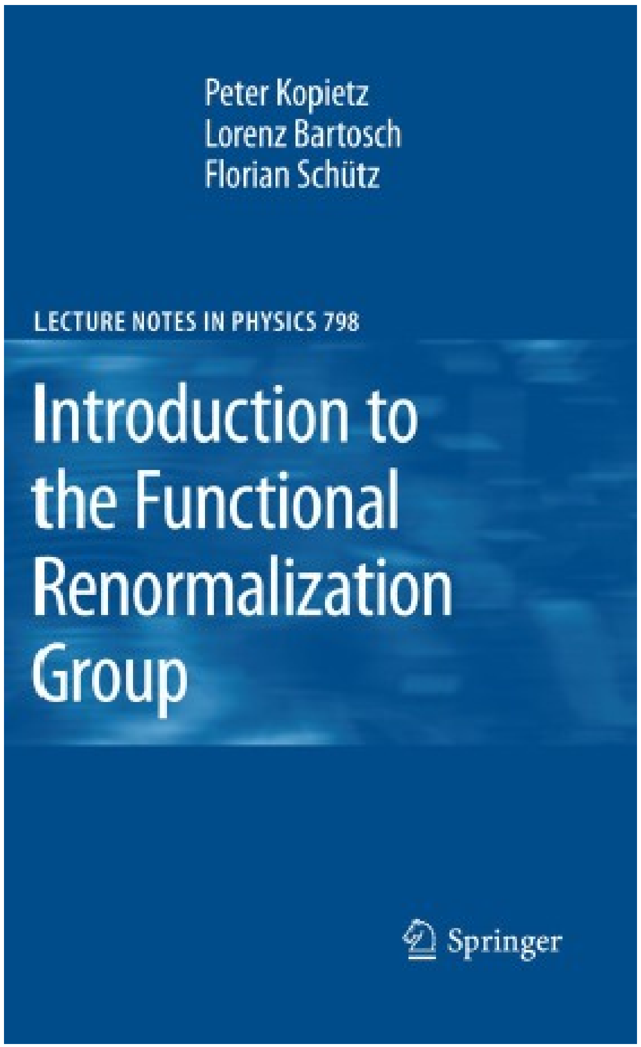
$$A(\tau;\omega) = \int_{-\infty}^{\infty} ds e^{i\omega s} [\hat{A}]_{\tau+\frac{s}{2},\tau-\frac{s}{2}}$$

$$\partial_\tau \text{Re} g(\tau;\omega) + 2(\omega - \epsilon_{\boldsymbol{k}}) \text{Im} g(\tau;\omega) \ + i\left(\hat{\Sigma}^M \hat{g} - \hat{g}^\dagger \hat{\Sigma}^M\right)_{(\tau;\omega)}$$

$$= \Sigma^{\text{in}}(\tau;\omega) - \Sigma^{\text{out}}(\tau;\omega)$$

Goal: get non-equilibrium self-energies from FRG!

2. Non-equilibrium functional renormalization group



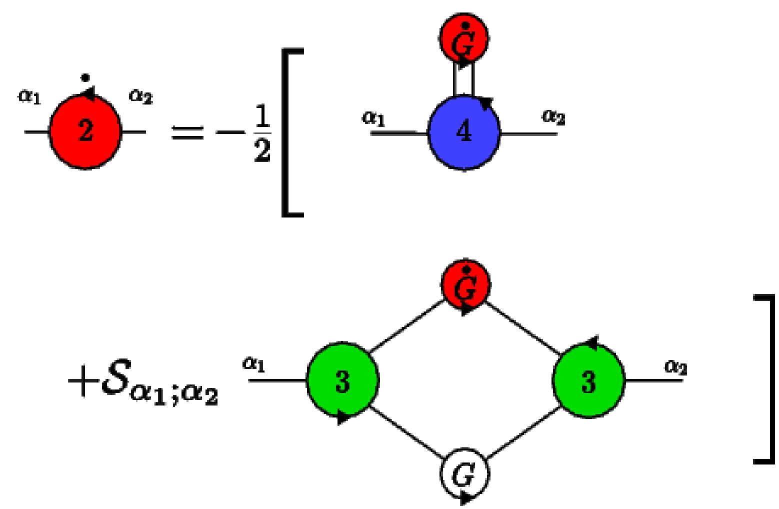
2010. XII, 380 p. (Lecture Notes in Physics, Vol. 798) Hardcover

- exact equation for change of generating functional of irreducible vertices as IR cutoff is reduced (Wetterich 1993)

$$\partial_\Lambda \Gamma_\Lambda[\Phi] = \frac{1}{2}\mathrm{Tr}\left[(\partial_\Lambda \boldsymbol{R}_\Lambda) \left(\frac{\delta}{\delta\Phi} \otimes \frac{\delta}{\delta\Phi} \Gamma_\Lambda[\Phi] + \boldsymbol{R}_\Lambda \right)^{-1} \right]$$

- exact RG flow equations for all vertices

- flow of self-energy:



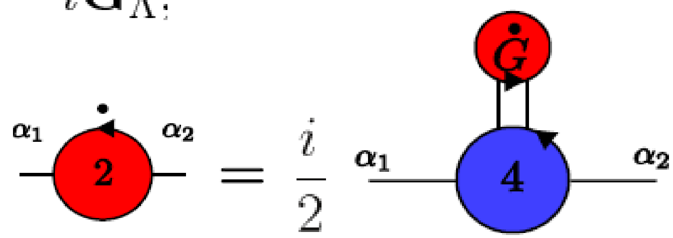
non-equilibrium FRG vertex expansion

(Gezzi et al, 2007; Gasenzer+Pawlowski, 2008; Kloss+PK 2010)

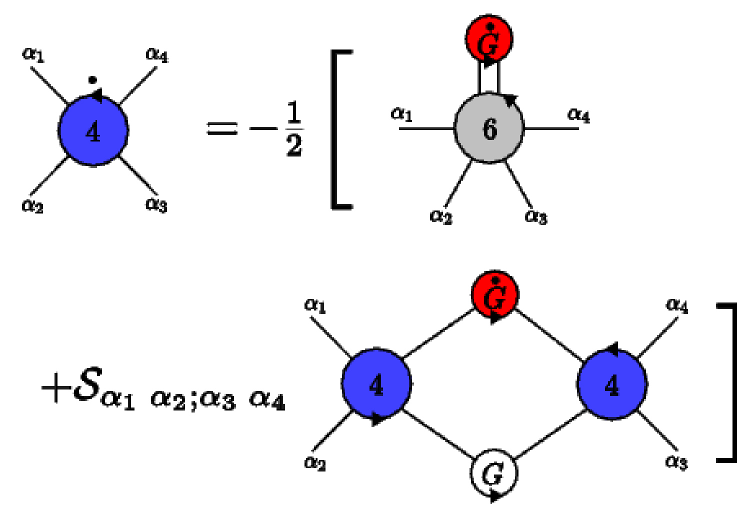
simple generalization of the equilibrium vertex expansion:

$$\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n} \to i\Gamma^{(n)}_{\Lambda,\alpha_1\ldots\alpha_n}; \qquad \mathbf{G}_\Lambda \to -i\mathbf{G}_\Lambda \;,\; \dot{\mathbf{G}}_\Lambda \to -i\dot{\mathbf{G}}_\Lambda;$$

$$\partial_\Lambda \Gamma^{(2)}_{\Lambda,\alpha_1\alpha_2} = \frac{i}{2}\int_{\beta_1}\int_{\beta_2} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2}\Gamma^{(4)}_{\Lambda,\beta_2\beta_1\alpha_1\alpha_2}$$



$$\partial_\Lambda \Gamma^{(4)}_{\Lambda,\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{i}{2}\int_{\beta_1}\int_{\beta_2} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2}\Gamma^{(6)}_{\Lambda,\beta_2\beta_1\alpha_1\alpha_2\alpha_3\alpha_4}$$
$$+\frac{i}{2}\int_{\beta_1}\int_{\beta_2}\int_{\beta_3}\int_{\beta_4} [\dot{\mathbf{G}}_\Lambda]_{\beta_1\beta_2}[\mathbf{G}_\Lambda]_{\beta_3\beta_4}$$
$$\times\Big[\Gamma^{(4)}_{\Lambda,\beta_2\beta_3\alpha_3\alpha_4}\Gamma^{(4)}_{\Lambda,\beta_4\beta_1\alpha_1\alpha_2} + \Gamma^{(4)}_{\Lambda,\beta_2\beta_3\alpha_1\alpha_2}\Gamma^{(4)}_{\Lambda,\beta_4\beta_3\alpha_3\alpha_4}$$
$$+(\alpha_1\leftrightarrow\alpha_2)+(\alpha_1\leftrightarrow\alpha_4)\Big]$$



non-equilibrium time-evolution from the FRG: the basic idea

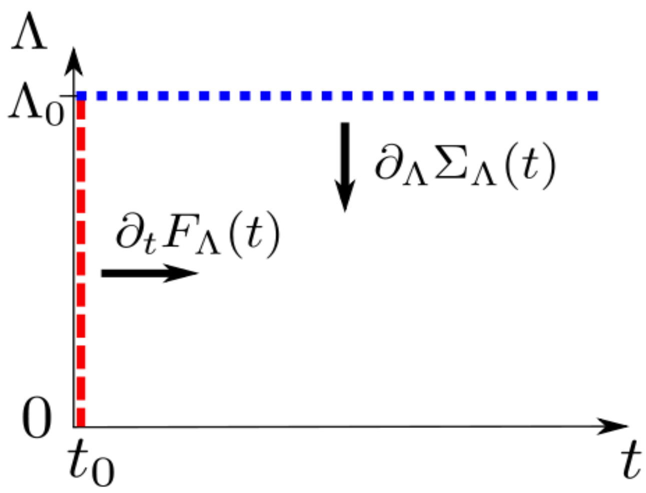
- introduce cutoff parameter Λ which somehow simplifies time evolution
- write down suitably truncated FRG flow equations for the self-energies
- structure of resulting equations:

$$\partial_\tau f_\Lambda(\boldsymbol{k},\omega,\tau) = C_\Lambda[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A]$$

$$\partial_\Lambda \Sigma^R_\Lambda(\boldsymbol{k},\omega,\tau) = I^R_\Lambda[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A]$$

$$\partial_\Lambda \Sigma^A_\Lambda(\boldsymbol{k},\omega,\tau) = I^A_\Lambda[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A],$$

$$\partial_\Lambda \Sigma^K_\Lambda(\boldsymbol{k},\omega,\tau) = I^K_\Lambda[\boldsymbol{k},\omega,\tau,f,\Sigma^K,\Sigma^R,\Sigma^A]$$



- make standard approximations to simplify system
  (e.g. reduction to Fokker-Planck eq)
  or solve by brute force numerically.

cutoff schemes for time-evolution

- cutoff should: 1. simplify time-evolution
  2. respect causality
  3. in equilibrium respect fluctuation-dissipation theorem

- proposals:

1. long-time cutoff (Gasenzer, Pawlowski, 2008)

2. out-scattering rate cutoff (Kloss, P.K., 2011)

$$G_0^R(\boldsymbol{k},\omega) = \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\eta} \to \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\Lambda}$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\Lambda} = -i\Theta(t) e^{-i\epsilon_{\boldsymbol{k}} t} e^{-\Lambda t}$$

Λ = artifical decay rate

….more cutoff schemes

3. hybridization cutoff (Jakobs, Pletyukhov, Schoeller, 2010)

$$G_0^R(\boldsymbol{k},\omega) = \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\eta} \to \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\Lambda} \qquad (\hat{G}_0^R)^{-1}\hat{G}_0^K(\hat{G}_0^A)^{-1} = 2i\eta\hat{f}_0 \to 2i\Lambda\hat{f}_0$$

Λ = hybridization energy due to coupling to external bath

4. bosonic hybridization cutoff (Hick, Kloss, PK, 2012)

$$G_0^R(\boldsymbol{k},\omega) = \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\eta} \to \frac{1}{\omega - \epsilon_{\boldsymbol{k}} + i\Lambda \mathrm{sgn}\omega} \qquad (\hat{G}_0^R)^{-1}\hat{G}_0^K(\hat{G}_0^A)^{-1} = 2i\eta\hat{f}_0 \to 2i\Lambda\hat{f}_0$$

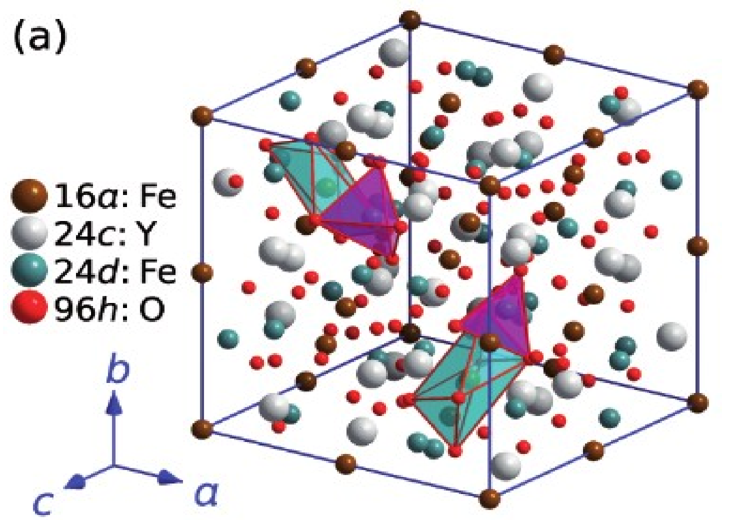
spectral function of bosons is negative for negative frequencies!

$$2\mathrm{Im}G^R(\boldsymbol{k},\omega) = -\rho(\boldsymbol{k},\omega) \qquad \rho(\boldsymbol{k},\omega) \begin{cases} \geq 0 & \text{for } \omega > 0 \\ \leq 0 & \text{for } \omega < 0 \end{cases}$$

3. Parametric resonance of magnons in yttrium-iron-garnet (YIG)

- what is YIG? ferromagnetic insulator

- at the first sight: too complicated!



◀ (a) Elementary cell of YIG with 160 atoms. The spins of the 16 Fe in positions a are coupled anti-ferromagnetically to the spins of the 24 in positions d and cause the ferrimagnetic ordering.

europhysicsnews HIGHLIGHTS FROM EUROPEANS JOURNALS

A. Kreisel, F. Sauli, L. Bartosch, PK, 2009

- effective quantum spin model for relevant magnon band:

$$\hat{H} = -\frac{1}{2}\sum_{ij} J_{ij}\boldsymbol{S}_i \cdot \boldsymbol{S}_j - \mu \boldsymbol{H}_e \cdot \sum_i \boldsymbol{S}_i - \frac{1}{2}\sum_{ij,i\neq j} \frac{\mu^2}{|\boldsymbol{R}_{ij}|^3}\left[3(\boldsymbol{S}_i \cdot \hat{\boldsymbol{R}}_{ij})(\boldsymbol{S}_j \cdot \hat{\boldsymbol{R}}_{ij}) - \boldsymbol{S}_i \cdot \boldsymbol{S}_j\right]$$

exchange interaction: $J = 1.29\ \mathrm{K}$.

lattice spacing: $a = 12.376\ \text{Å}$

saturation magnetization: $4\pi M_S = 1750\ \mathrm{G}$

effective spin: $S = M_s a^3/\mu \approx 14.2$

experiments on YIG: probing the non-equilibrium dynamics of magnons

•motivation:

collaboration with experimental group of B. Hillebrands (Kaiserslautern)

non-equilibrium dynamics of interacting magnons in YIG

nature Vol 443|28 September 2006|doi:10.1038/nature05117

LETTERS

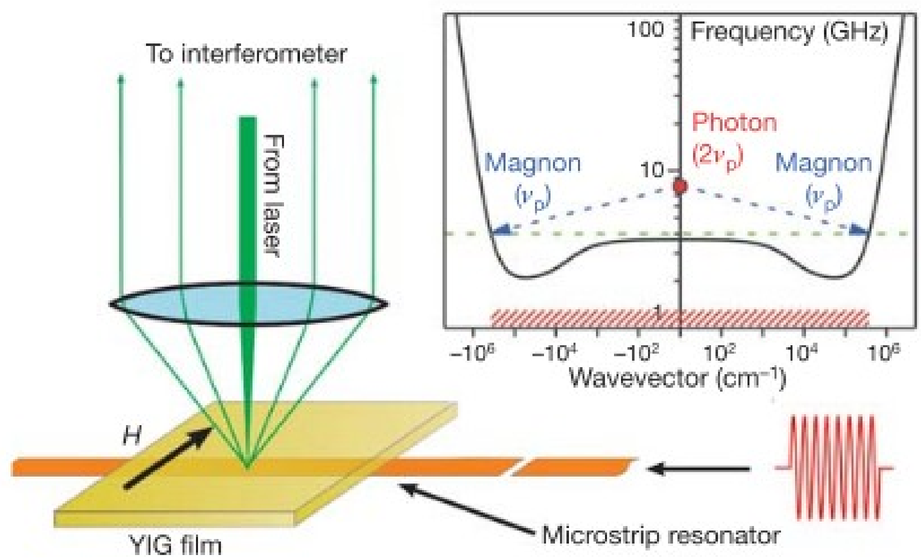
Bose–Einstein condensation of quasi-equilibrium magnons at room temperature under pumping

S. O. Demokritov[1], V. E. Demidov[1], O. Dzyapko[1], G. A. Melkov[2], A. A. Serga[3], B. Hillebrands[3] & A. N. Slavin[4]

•experiment:

microwave-pumping of magnons in YIG

measurement of magnon distriubution via Brillouin light scattering



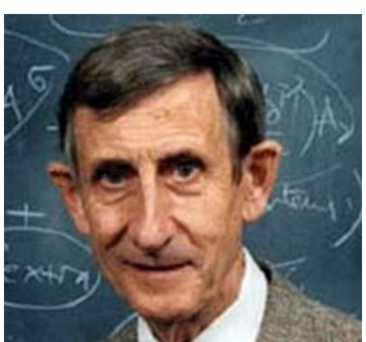
from spin operators to bosons

- **problem:** spin-algebra is very complicated: $[S_i^\alpha, S_j^\beta] = i\delta_{ij}\epsilon^{\alpha\beta\gamma}S_i^\gamma \quad \boldsymbol{S}_i^2 = S(S+1)$
- **solution:** for ordered magnets: bosonization of spins (Holstein,Primakoff 1940)

$$S_i^+ = S_i^x + iS_i^y = \sqrt{2S}\sqrt{1-\frac{b_i^\dagger b_i}{2S}}\, b_i = \sqrt{2S}\left[b_i - \frac{b_i^\dagger b_i b_i}{4S} + \dots\right]$$

$$S_i^z = S - b_i^\dagger b_i$$

- spin algebra indeed satisfied if $[b_i, b_j^\dagger] = \delta_{ij}$
- proof that different dimension of Hilbert spaces does not matter by Dyson 1956:



PHYSICAL REVIEW VOLUME 102, NUMBER 5 JUNE 1, 1956

General Theory of Spin-Wave Interactions*

FREEMAN J. DYSON

Department of Physics, University of California, Berkeley, California, and Institute for Advanced Study, Princeton, New Jersey

(Received February 2, 1956)

some history: magnon dynamics in YIG

H. Suhl, 1957, E. Schlömann et al, 1960s,
V. E. Zakharov, V. S. L'vov, S. S. Starobinets, 1970s

- minimal model:

$$\hat{H}_{\mathrm{res}}(t) = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + \frac{1}{2}\sum_{\boldsymbol{k}} \left[\gamma_{\boldsymbol{k}} e^{-i\omega_0 t} a^\dagger_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}} + \gamma^*_{\boldsymbol{k}} e^{i\omega_0 t} a_{-\boldsymbol{k}} a_{\boldsymbol{k}}\right] + \frac{1}{2}\sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}} u(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{q}) a^\dagger_{\boldsymbol{k}+\boldsymbol{q}} a^\dagger_{\boldsymbol{k}'-\boldsymbol{q}} a_{\boldsymbol{k}'} a_{\boldsymbol{k}}$$

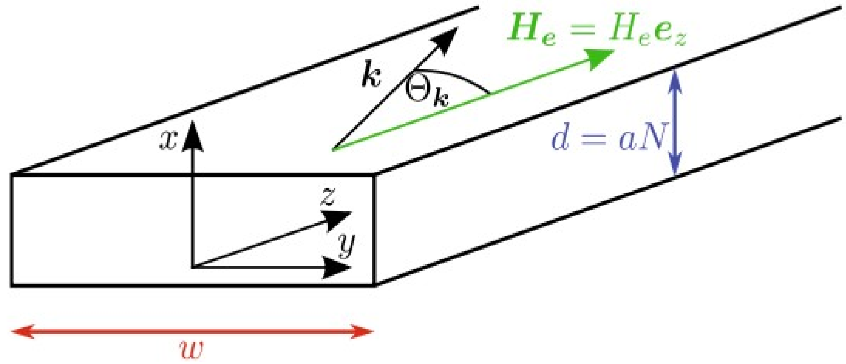
- "S-theory": time-dependent self-consistent Hartree-Fock approximation for magnon distributions functions

$$n_{\boldsymbol{k}}(t) = \langle a^\dagger_{\boldsymbol{k}}(t) a_{\boldsymbol{k}}(t)\rangle$$
$$p_{\boldsymbol{k}}(t) = \langle a_{-\boldsymbol{k}}(t) a_{\boldsymbol{k}}(t)\rangle$$

weak points:
- no microscopic description of dissipation and damping
- possibility of BEC not included!

- goals:
  - consistent quantum kinetic theory for magnons in YIG beyond Hartree-Fock
  - include time-evolution of Bose-condendsate
  - develop functional renormalization group for non-equilibrium

magnon dispersion of finite YIG films



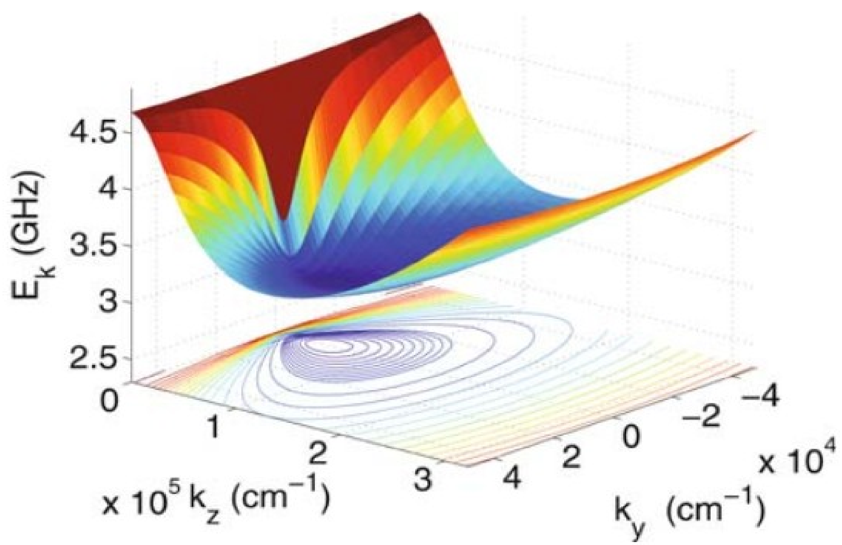
$H_e \approx 1000$ Oe

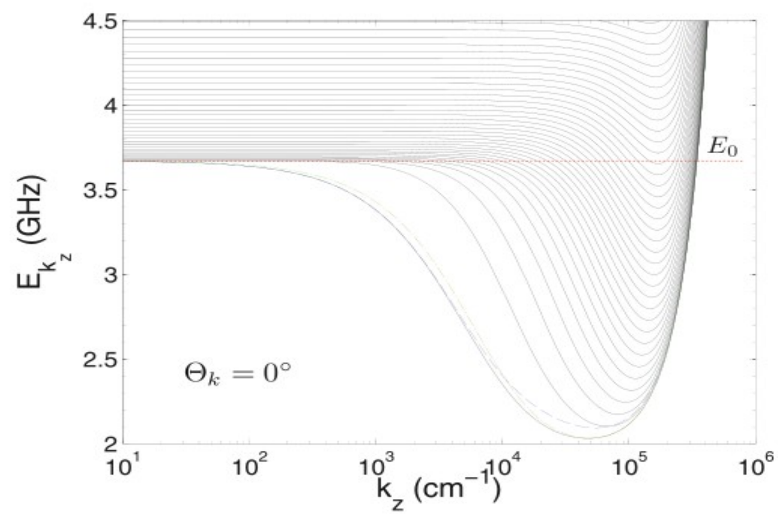
$d \approx 5 \mu m \approx 4000a$

$w \gg d$

- dispersion of lowest magnon mode has minimum at finite k

due to interplay between: 1. exchange interaction
2. dipole-dipole interaction
3. finite width of films





toy model for dynamics of lowest magnon mode

T. Kloss, A. Kreisel, PK, PRB 2010

- keep only lowest magnon mode

→ anharmonic oscillator with off-diagonal pumping:

$$\hat{H}(t) = \epsilon_0 a^\dagger a + \frac{\gamma_0}{2} e^{-i\omega_0 t} a^\dagger a^\dagger + \frac{\gamma_0^*}{2} e^{i\omega_0 t} aa + \frac{u}{2} a^\dagger a^\dagger aa$$

- rotating reference frame: $\tilde{a} = e^{\frac{i}{2}\omega_0 t} a$

$$\tilde{H} = \tilde{\epsilon}_0 \tilde{a}^\dagger \tilde{a} + \frac{\gamma_0}{2} \tilde{a}^\dagger \tilde{a}^\dagger + \frac{\gamma_0^*}{2} \tilde{a}\tilde{a} + \frac{u}{2} \tilde{a}^\dagger \tilde{a}^\dagger \tilde{a}\tilde{a} \qquad \tilde{\epsilon}_0 = \epsilon_0 - \frac{\omega_0}{2}$$

- instability of non-interacting system for large pumping:

$$\tilde{a} = \frac{\hat{X} + i\hat{P}}{\sqrt{2}} \qquad \tilde{a}^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}}$$

$$\tilde{\epsilon}_0 \tilde{a}^\dagger \tilde{a} + \frac{\gamma_0}{2} [\tilde{a}^\dagger \tilde{a}^\dagger + \tilde{a}\tilde{a}] = \frac{\tilde{\epsilon}_0 - \gamma_0}{2} \hat{P}^2 + \frac{\tilde{\epsilon}_0 + \gamma_0}{2} \hat{X}^2.$$

  - for $\gamma_0 > |\tilde{\epsilon}_0|$ oscillator has negative mass
  - non-interacting hamiltonian not positive definite

→ parametric resonance

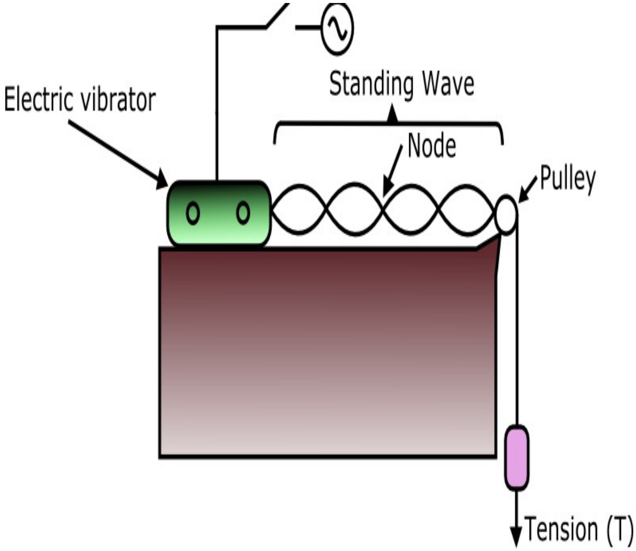
parametric resonance

- what is parametric resonance?
  - classical harmonic oscillator with harmonic frequency modulation:

$$\frac{d^2x(t)}{dt^2} + \Omega^2(t)x(t) = 0 \qquad \Omega(t) = \Omega_0 + \Omega_1 \cos(\omega_0 t)$$

  - resonance condition: $\omega_0 \approx 2\Omega_0$

→ oscillator absorbs energy at a rate proportional to the energy it already has!



- history:
  - discovered: Melde experiment, 1859

    excite oscillations of string by periodically varying its tension at twice its resonance frequency
  - theoretically explained: Rayleigh 1883

time-dependent Hartree-Fock approximation ("S-theory")

• order parameter $\phi(t) \equiv \langle \tilde{a}(t) \rangle$ Gross-Pitaevskii equation:

$$i\partial_t \phi = \tilde{\epsilon}_c(t)\phi + \gamma_c(t)\phi^* + u|\phi|^2\phi$$

$$\tilde{\epsilon}_c(t) = \tilde{\epsilon}_0 + 2un_c(t)$$

$$\gamma_c(t) = \gamma_0 + u\tilde{p}_c(t)$$

• connected correlation functions: $n_c(t) = \langle \delta\tilde{a}^\dagger(t)\delta\tilde{a}(t) \rangle \quad \tilde{p}_c(t) = \langle \delta\tilde{a}(t)\delta\tilde{a}(t) \rangle$

$$\delta\tilde{a}(t) = \tilde{a}(t) - \langle \tilde{a}(t) \rangle$$

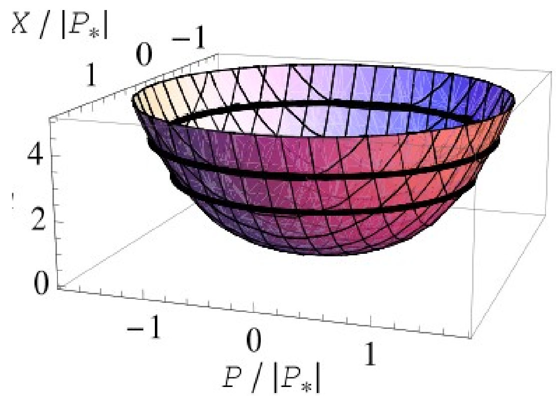
kinetic equations:

$$\begin{aligned} i\partial_t n_c(t) &= \gamma(t)\tilde{p}_c^*(t) - \gamma^*(t)\tilde{p}_c(t), \\ i\partial_t \tilde{p}_c(t) &= 2\tilde{\epsilon}(t)\tilde{p}_c(t) + \gamma(t)[2n_c(t)+1] \end{aligned}$$

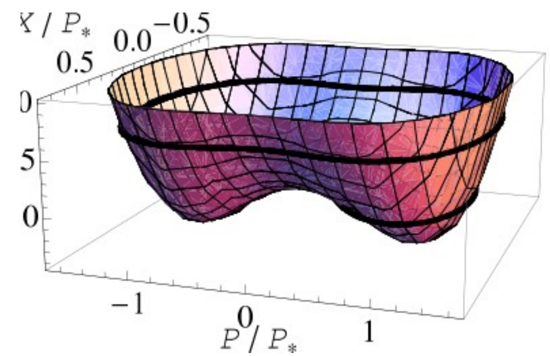
$$\begin{aligned} \tilde{\epsilon}(t) &= \tilde{\epsilon}_0 + 2u[n_c(t) + |\phi(t)|^2] \\ \gamma(t) &= \gamma_0 + u[\tilde{p}_c(t) + \phi^2(t)] \end{aligned}$$

• order parameter: Hamiltonian dynamics in effective potential (Hartree-Fock)

$|\gamma_0| < |\tilde{\epsilon}_0|$



$|\gamma_0| > |\tilde{\epsilon}_0|$



4. Time-evolution of the toy model from the FRG

(T. Kloss, P.K., Phys. Rev. B 2011)

$$\mathcal{H}(t) = \epsilon a^\dagger a + \frac{1}{2}\left[\gamma e^{-i\omega_0 t} a^\dagger a^\dagger + \gamma^* e^{i\omega_0 t} a a\right]$$

- toy model can be solved numerically exactly by solving time-dependent Schrödinger equation

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n(t)|n\rangle$$

$$i\hbar\partial_t \psi_n(t) = \left[\epsilon_0 n + \frac{u_0}{2} n(n-1)\right]\psi_n(t) + \frac{\gamma_0}{2} e^{-i\omega_0 t}\sqrt{n(n-1)}\psi_{n-2}(t) + \frac{\gamma_0^*}{2} e^{i\omega_0 t}\sqrt{(n+2)(n+1)}\psi_{n+2}(t)$$

- generalize FRG to include also off-diagonal Green functions:

$$\begin{aligned} g^R(t,t') &= -i\Theta(t-t')\langle[a(t),a^\dagger(t')]\rangle \\ g^A(t,t') &= i\Theta(t'-t)\langle[a(t),a^\dagger(t')]\rangle \\ g^K(t,t') &= -i\langle\{a(t),a^\dagger(t')\}\rangle \end{aligned}$$

$$\begin{aligned} p^R(t,t') &= -i\Theta(t-t')\langle[a(t),a(t')]\rangle \\ p^A(t,t') &= i\Theta(t'-t)\langle[a(t),a(t')]\rangle \\ p^K(t,t') &= -i\langle\{a(t),a(t')\}\rangle \end{aligned}$$

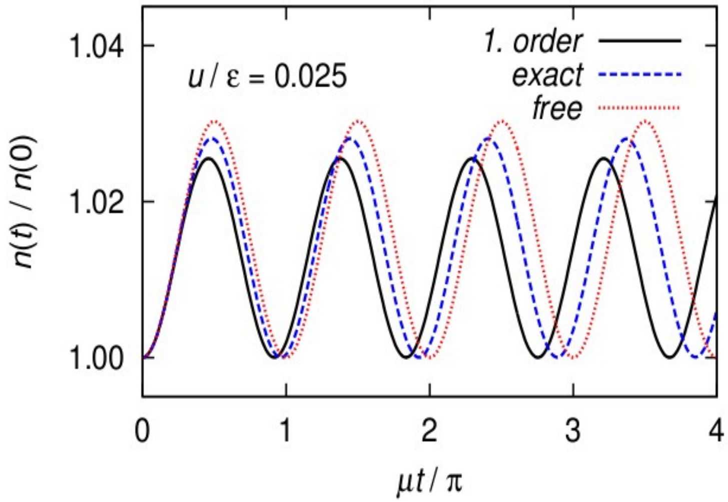
- Keldysh component at equal times gives distribution functions:

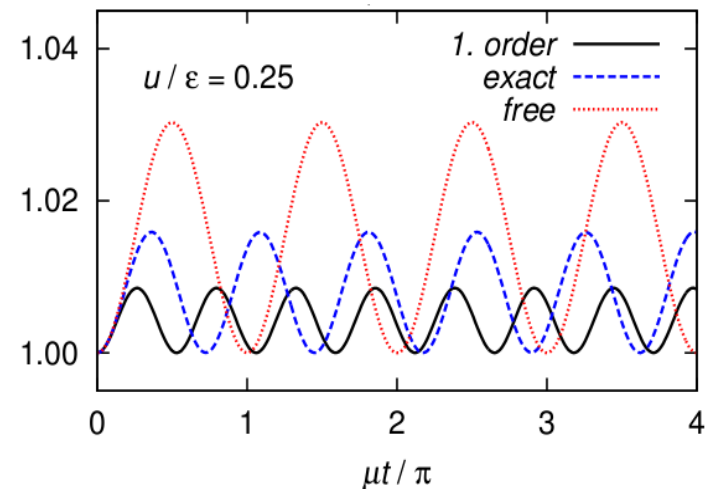
$$G^K(t,t) = \begin{pmatrix} p^K(t,t) & g^K(t,t) \\ g^K(t,t) & p^K(t,t)^* \end{pmatrix} = -2i\begin{pmatrix} p(t) & n(t)+\frac{1}{2} \\ n(t)+\frac{1}{2} & p^*(t) \end{pmatrix}$$

time-dependent Hartree-Fock: comparison with exact non-equilibrium dynamics

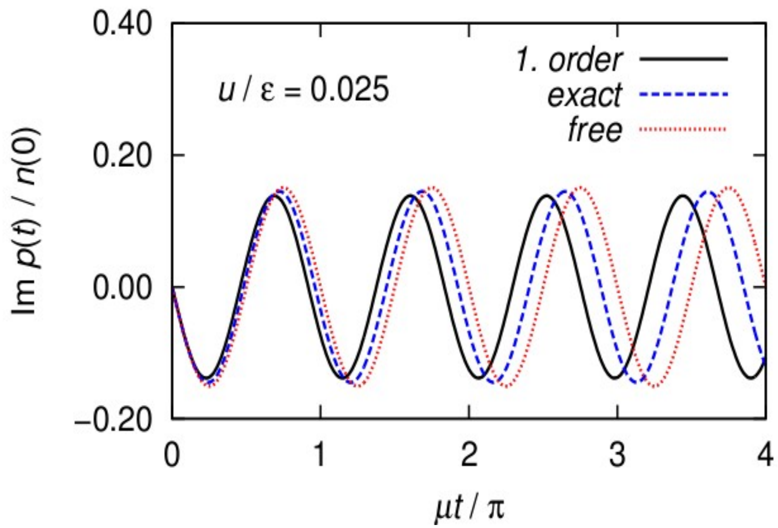
small interaction: larger interaction:

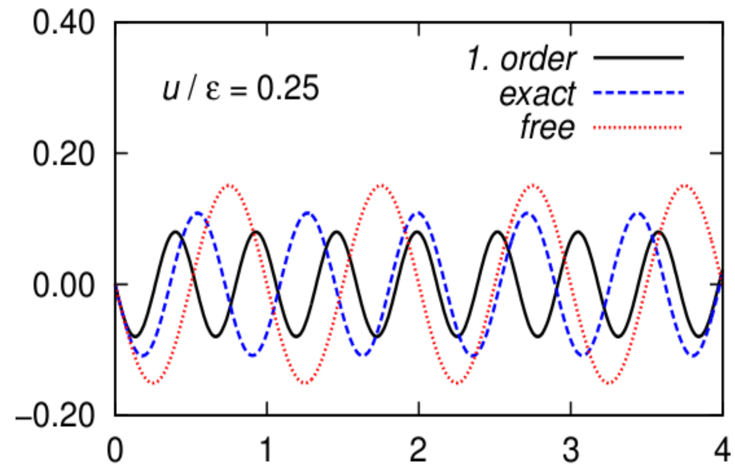
diagonal correlator:





off-diagonal correlator:

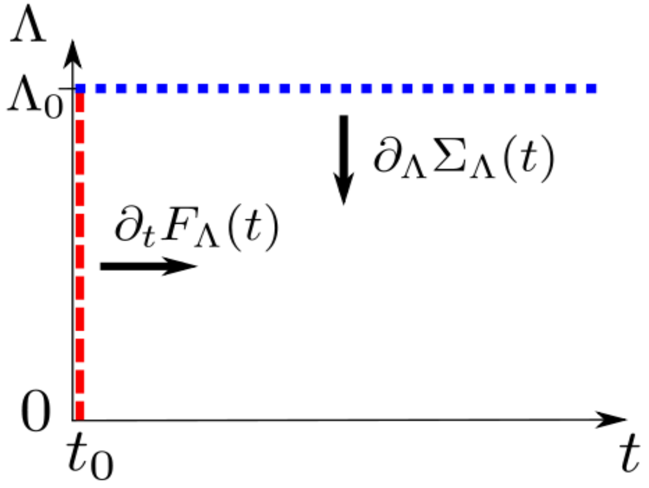




first order truncation of the FRG hierarchy

$$i\partial_t F_\Lambda(t) = -M_\Lambda^T(t)F_\Lambda(t) - F_\Lambda(t)M_\Lambda(t)$$

$$\partial_\Lambda \Sigma_\Lambda(t) = iu \begin{pmatrix} \frac{1}{2}\dot{G}^K_{\Lambda,\bar{a}\bar{a}}(t,t) & \dot{G}^K_{\Lambda,a\bar{a}}(t,t) \\ \dot{G}^K_{\Lambda,\bar{a}a}(t,t) & \frac{1}{2}\dot{G}^K_{\Lambda,aa}(t,t) \end{pmatrix}$$



$$F_\Lambda(t) = \begin{pmatrix} -2p^*_\Lambda(t) & 2n_\Lambda(t)+1 \\ 2n_\Lambda(t)+1 & -2p_\Lambda(t) \end{pmatrix}$$

$$\dot{G}^K_\Lambda(t,t) \approx 2i \int_{t_0}^{t} dt_1 G^R_{0,\Lambda}(t,t_1)F_\Lambda(t)G^A_{0,\Lambda}(t_1,t)$$

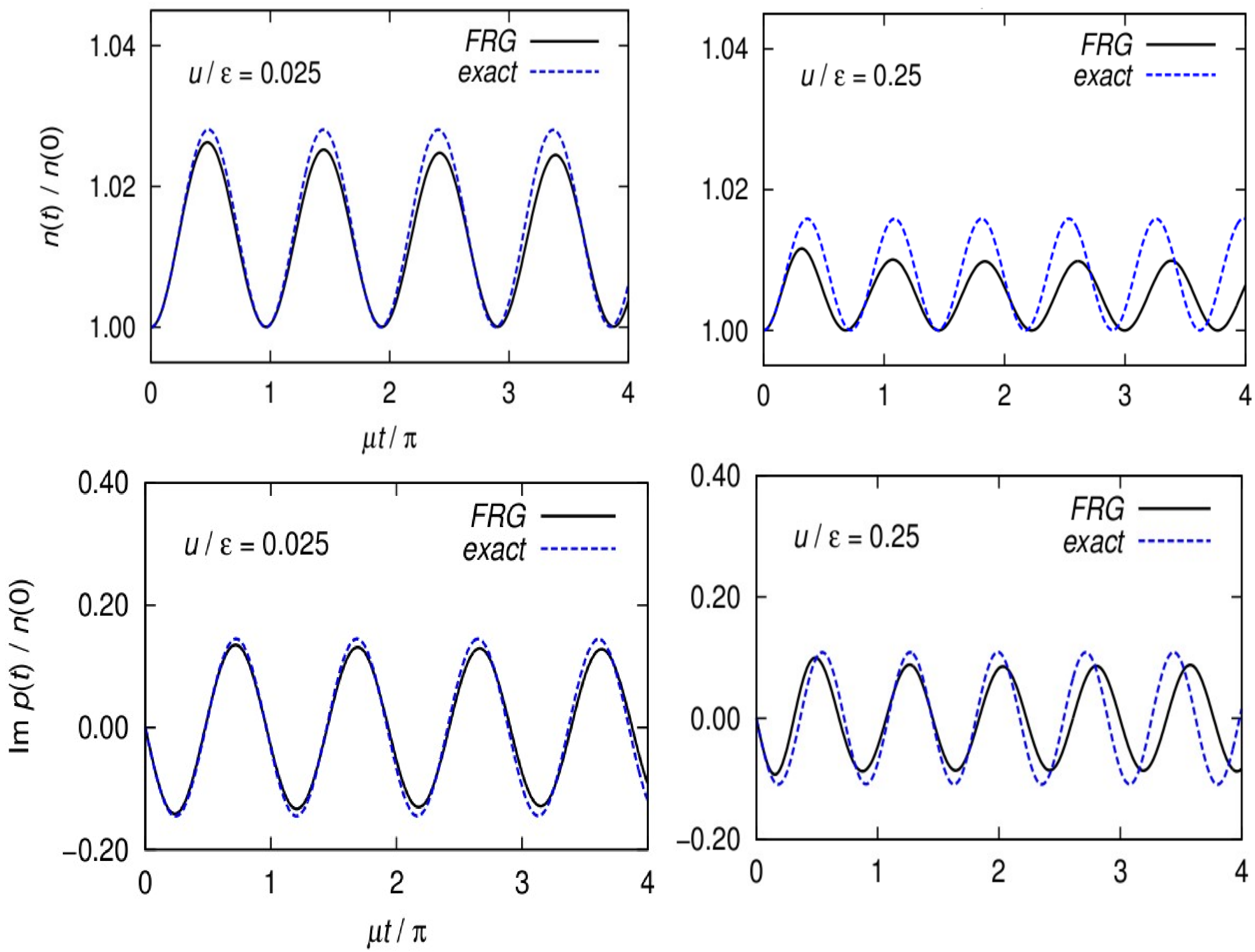
$$M_\Lambda(t) = M - i\Lambda I + Z\Sigma_\Lambda(t) = \begin{pmatrix} \epsilon - u - i\Lambda + \Sigma_{\Lambda,\bar{a}a}(t) & |\gamma| + \Sigma_{\Lambda,\bar{a}\bar{a}}(t) \\ -|\gamma| - \Sigma_{\Lambda,aa}(t) & -[\epsilon - u + i\Lambda + \Sigma_{\Lambda,a\bar{a}}(t)] \end{pmatrix}$$

comparison of first order FRG with exact non-equilibrium dynamics

small interaction: larger interaction:

diagonal correlator:

off-diagonal correlator:



3. Thermalization of Magnons in Yttrium-Iron Garnet

J. Hick, T. Kloss, PK, Phys. Rev. B 2013

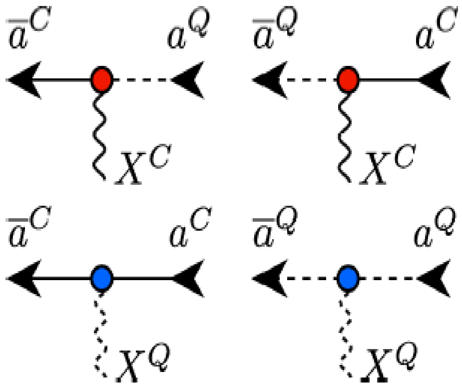
$$\mathcal{H} = \sum_{\boldsymbol{k}} \epsilon_{\boldsymbol{k}} a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}} + \sum_{\boldsymbol{q}} \omega_{\boldsymbol{q}} b_{\boldsymbol{q}}^\dagger b_{\boldsymbol{q}} + \frac{1}{\sqrt{V}} \sum_{\boldsymbol{q}} \gamma_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} (b_{\boldsymbol{q}} + b_{-\boldsymbol{q}}^\dagger)$$

magnon phonon

- cutoff-procedure for FRG: Keldysh-action with hybridization cutoff only in the phonons:

$$X_{\boldsymbol{q}} = \frac{1}{\sqrt{2}} \left(b_{\boldsymbol{q}} + b_{-\boldsymbol{q}}^\dagger \right)$$

$$S_\Lambda[\bar{a}, a, \bar{b}, b] = \int dt \int dt' \Bigg\{ \sum_{\boldsymbol{k}} (\bar{a}_{\boldsymbol{k}}^C(t), \bar{a}_{\boldsymbol{k}}^Q(t)) \begin{pmatrix} 0 & (\hat{G}_0^A)^{-1} \\ (\hat{G}_0^R)^{-1} & 2i\eta\hat{g}_0 \end{pmatrix}_{tt'} \begin{pmatrix} a_{\boldsymbol{k}}^C(t') \\ a_{\boldsymbol{k}}^Q(t') \end{pmatrix}$$
$$+ \sum_{\boldsymbol{q}} (\bar{b}_{\boldsymbol{q}}^C(t), \bar{b}_{\boldsymbol{q}}^Q(t)) \begin{pmatrix} 0 & (\hat{F}_{0,\Lambda}^A)^{-1} \\ (\hat{F}_{0,\Lambda}^R)^{-1} & 2i\Lambda\hat{f}_0 \end{pmatrix}_{tt'} \begin{pmatrix} b_{\boldsymbol{q}}^C(t') \\ b_{\boldsymbol{q}}^Q(t') \end{pmatrix} \Bigg\}$$
$$- \int dt \frac{1}{\sqrt{V}} \sum_{\boldsymbol{k},\boldsymbol{q}} \gamma_{\boldsymbol{q}} \left[\left(\bar{a}_{\boldsymbol{k}+\boldsymbol{q}}^C a_{\boldsymbol{k}}^Q + \bar{a}_{\boldsymbol{k}+\boldsymbol{q}}^Q a_{\boldsymbol{k}}^C \right) X_{\boldsymbol{q}}^C + \left(\bar{a}_{\boldsymbol{k}+\boldsymbol{q}}^C a_{\boldsymbol{k}}^C + \bar{a}_{\boldsymbol{k}+\boldsymbol{q}}^Q a_{\boldsymbol{k}}^Q \right) X_{\boldsymbol{q}}^Q \right]$$



rate equation for magnon distribution

(Banyai et al, 2000)

$$\partial_t n_{\boldsymbol{k}}(t) = \frac{1}{V}\sum_{\boldsymbol{k}'}\Big\{[1+n_{\boldsymbol{k}}(t)]W_{\boldsymbol{k},\boldsymbol{k}'}n_{\boldsymbol{k}'}(t) - [1+n_{\boldsymbol{k}'}(t)]W_{\boldsymbol{k}',\boldsymbol{k}}n_{\boldsymbol{k}}(t)\Big\}$$

- Transition rates from Fermi`s golden rule (phonons in equilibrium):

$$W_{\boldsymbol{k},\boldsymbol{k}'} = 2\gamma^2_{\boldsymbol{k}-\boldsymbol{k}'}b(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})D^I_{\boldsymbol{k}-\boldsymbol{k}'}(\epsilon_{\boldsymbol{k}}-\epsilon_{\boldsymbol{k}'})$$

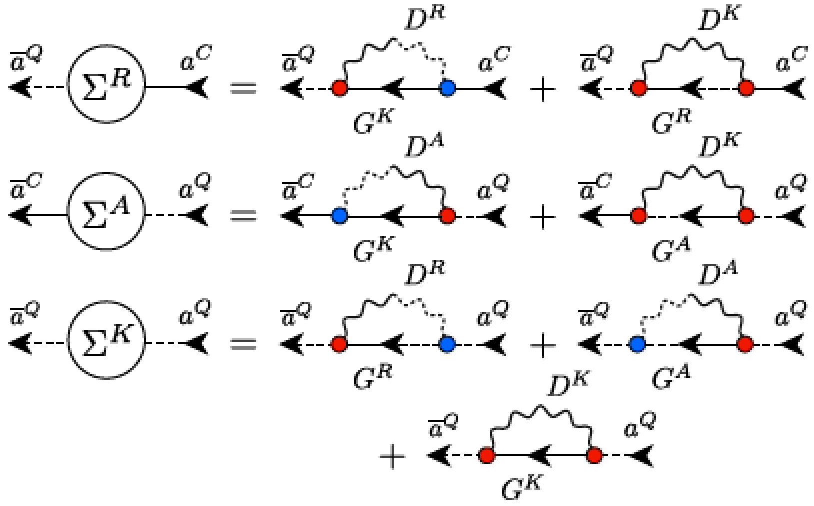
$$b(\omega) = \frac{1}{e^{\beta\omega}-1} \qquad D^I_{\boldsymbol{q}}(\omega) = \pi\,\mathrm{sgn}(\omega)\left[\delta(\omega-\omega_{\boldsymbol{q}})+\delta(\omega+\omega_{\boldsymbol{q}})\right]$$

- Detailed balance: $W_{\boldsymbol{k},\boldsymbol{k}'}e^{-\beta\epsilon_{\boldsymbol{k}'}} = W_{\boldsymbol{k}',\boldsymbol{k}}e^{-\beta\epsilon_{\boldsymbol{k}}}$

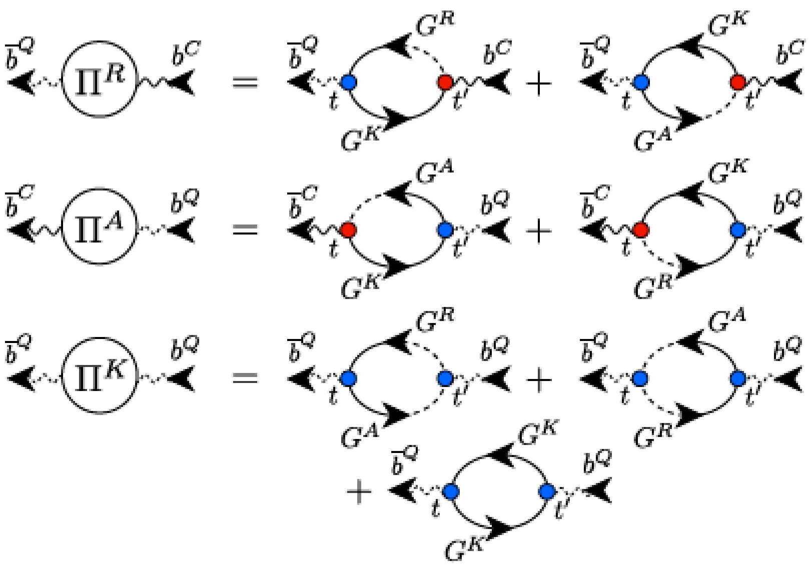
- Number conservation: $\frac{\partial}{\partial t}\sum_{\boldsymbol{k}} n_{\boldsymbol{k}}(t) = 0$

underlying approximations

1. Magnon self-energies to second order in magnon-phonon coupling



2. Neglect phonon self-energies:



….underlying approximations, continued

3. Keep only leading terms in gradient expansion (factorization of Wigner transforms of products)

$$[\hat{A}\hat{B}]_{(\tau;\omega)} = \int ds_1 \int ds_2 \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi} e^{i(\omega_1 s_2 - \omega_2 s_1)} A(\tau + \frac{s_1}{2}; \omega + \omega_1) B(\tau + \frac{s_2}{2}; \omega + \omega_2)$$

$$\approx A(\tau;\omega)B(\tau;\omega) + \frac{1}{2i}\left[\partial_\tau A(\tau;\omega)\partial_\omega B(\tau;\omega) - \partial_\omega A(\tau;\omega)\partial_\tau B(\tau;\omega)\right]$$

4. Neglect renormalization of real part of magnon energies and take frequency argument of self-energies on resonance. (quasiparticle approximation).

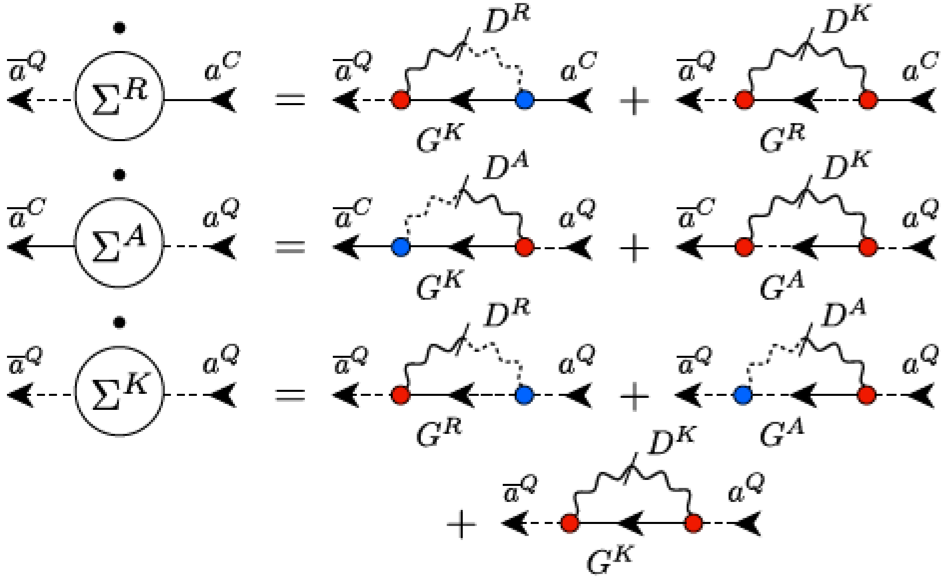
FRG rate equations for magnon distribution

- Hybridization cutoff only on phonons.
- Phonon damping due to magnons included.

$$F^R_{\Lambda,\boldsymbol{q}}(\tau;\omega) = \frac{1}{\omega - \omega_{\boldsymbol{q}} + i\Lambda \mathrm{sgn}\omega + \frac{i}{2}\Pi^I_{\Lambda,\boldsymbol{q}}(\tau;\omega)} \qquad iF^K_{\Lambda;\boldsymbol{q}}(\tau,\omega) = \coth\left(\frac{\beta\omega}{2}\right) F^I_{\Lambda,\boldsymbol{q}}(\tau;\omega).$$

$$\Pi^{\mathrm{I}}_{\Lambda,\boldsymbol{q}}(\tau;\omega) = \frac{2\pi}{V}\gamma^2_{\boldsymbol{q}} \sum_{\boldsymbol{k}} \delta\left(\omega + \epsilon_{\boldsymbol{k}-\boldsymbol{q}} - \epsilon_{\boldsymbol{k}}\right) \left\{ n_{\Lambda,\boldsymbol{k}-\boldsymbol{q}}(\tau) - n_{\Lambda,\boldsymbol{k}}(\tau) \right\}$$

- Truncated FRG flow equations for magnon self-energies.



...FRG rate equations, continued

- Kinetic equation for scale-dependent magnon distribution:

$$[\partial_\tau + \mu_\Lambda(\tau)] 2 n_{\Lambda,\boldsymbol{k}}(\tau) = i\Sigma^K_{\Lambda,\boldsymbol{k}}(\tau;\epsilon_{\boldsymbol{k}}) - \Sigma^I_{\Lambda,\boldsymbol{k}}(\tau;\epsilon_{\boldsymbol{k}}) \left[1 + 2n_{\Lambda,\boldsymbol{k}}(\tau)\right]$$

- FRG equations for non-equilibrium self-energies:

$$\partial_\Lambda i\Sigma^K_{\Lambda,\boldsymbol{k}}(\tau;\epsilon_{\boldsymbol{k}}) = \frac{1}{V}\sum_{\boldsymbol{k}'}\{[1 + n_{\Lambda,\boldsymbol{k}'}(\tau)]\dot{W}_{\boldsymbol{k}',\boldsymbol{k}} + \dot{W}_{\boldsymbol{k},\boldsymbol{k}'} n_{\Lambda,\boldsymbol{k}'}(\tau)\}$$

$$\partial_\Lambda \Sigma^I_{\Lambda,\boldsymbol{k}}(\tau;\epsilon_{\boldsymbol{k}}) = \frac{1}{V}\sum_{\boldsymbol{k}'}\{[1 + n_{\Lambda,\boldsymbol{k}'}(\tau)]\dot{W}_{\boldsymbol{k}',\boldsymbol{k}} - \dot{W}_{\boldsymbol{k},\boldsymbol{k}'} n_{\Lambda,\boldsymbol{k}'}(\tau)\}$$

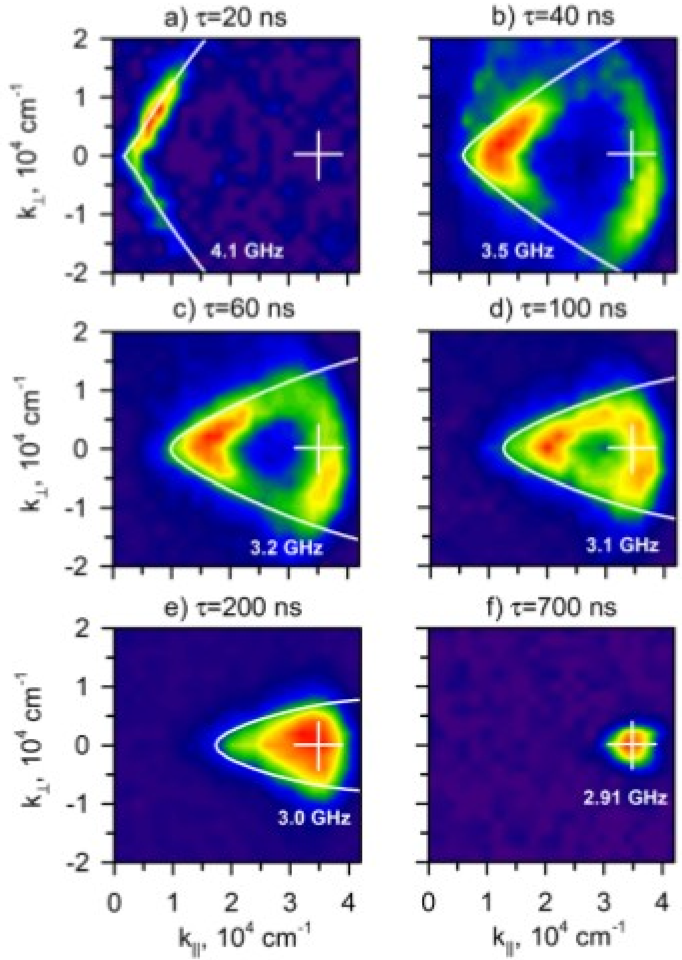
- Change of transition rates: $\dot{W}_{\boldsymbol{k},\boldsymbol{k}'} = 2\gamma^2_{\boldsymbol{k}-\boldsymbol{k}'} \dot{D}^I_{\Lambda,\boldsymbol{k}-\boldsymbol{k}'}(\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}'}) b(\epsilon_{\boldsymbol{k}} - \epsilon_{\boldsymbol{k}'})$

- Langrange multiplier to implement constant particle number:

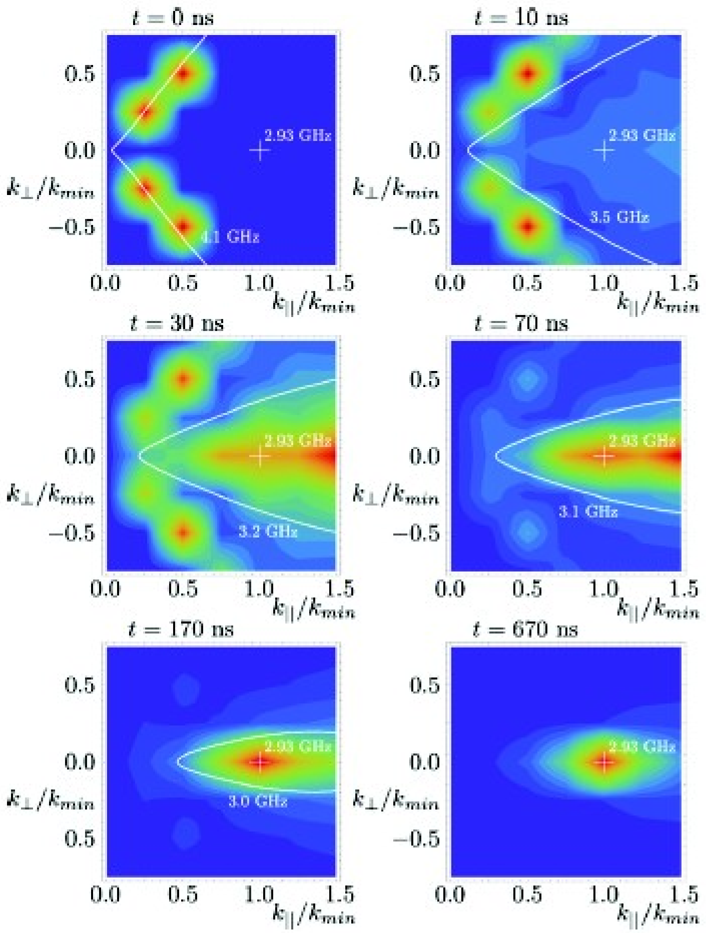
$$\mu_\Lambda(\tau) = \frac{1}{2N}\sum_{\boldsymbol{k}}\left\{ i\Sigma^K_{\Lambda,\boldsymbol{k}}(\tau;\epsilon_{\boldsymbol{k}}) - \Sigma^I_{\Lambda,\boldsymbol{k}}(\tau;\epsilon_{\boldsymbol{k}}) \left[1 + 2n_{\Lambda,\boldsymbol{k}}(\tau)\right] \right\}$$

comparison with experiments:

Experiment:
Demidov et al., Phys. Rev. Lett. (2008)



Our FRG calculation:



Summary+Outlook

- FRG approach to non-equilibrium dynamics of bosons
- cutoff procedures: hybridization cutoff in phonons
- good agreement with experiment:
  magnons in YIG are good quasiparticles

Outlook:

- quantum kinetics of BEC from the FRG
- include two-body interactions
- parametric resonance in YIG: beyond S-theory.