Kolloquium Universität Innsbruck

October 13, 2009

The renormalization group: from the foundations to modern applications

Peter Kopietz, Universität Frankfurt

1.) Historical introduction: what is the RG?

2.) The basic idea of the Wilsonian RG

3.) Modern formulation: functional RG

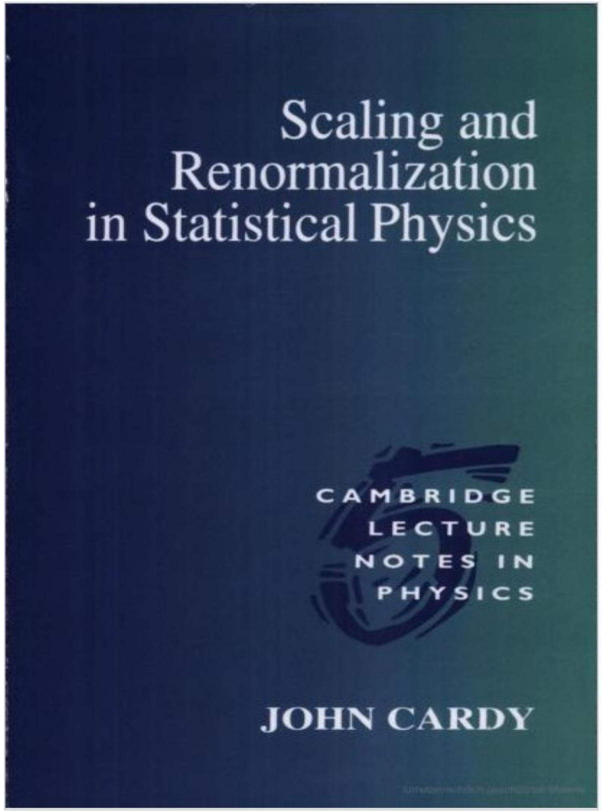
4.) Application: BCS-BEC crossover

1.Historical introduction:

What is the renormalization group?

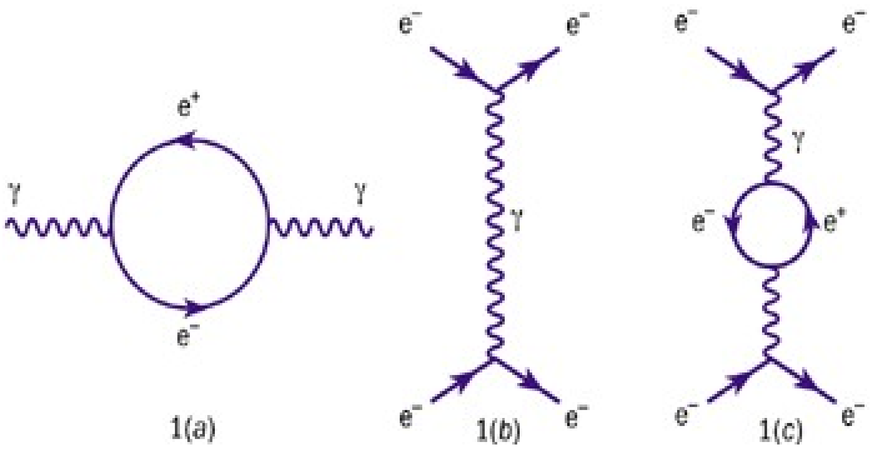
“... the renormalization group is merely a framework, a set of ideas, which has to be adapted to the problem at hand...

All renormalization group studies have in common the idea of re-expressing the parameters which define a problem in terms of some other, perhaps simpler set, while keeping unchanged those physical aspects of a problem which are of interest.”
(J. Cardy, 1996)



origin of renormalization: quantum field theory, 1940s

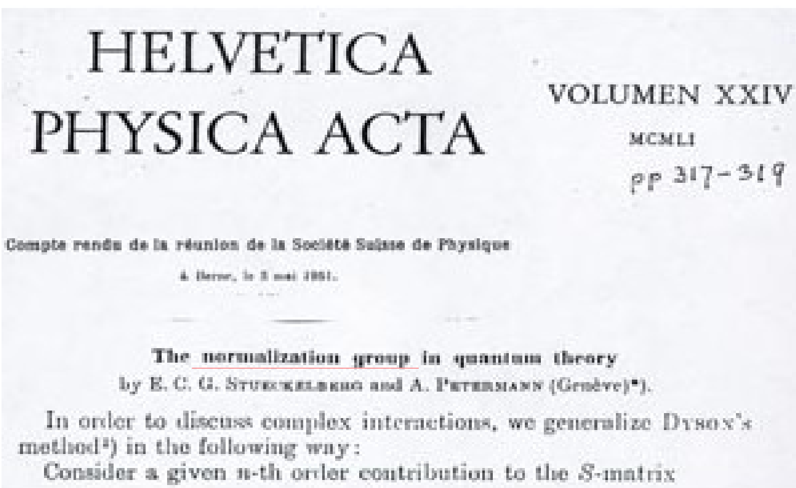
- problem: perturbation theory in quantum electrodynamics gives rise to infinite terms:



- solution: (Bethe, Feynman, Schwinger, Dyson, 1940s)
  - all infinities can be absorbed in redefinition (=renormalization) of a finite number of parameters (masses, coupling constants)
  - physical quantities can be expressed in terms of finite renormalized couplings (bare couplings are infinite)

origin of the RG: Stueckelberg and Petermann, 1951

- understand high-energy behavior of renormalized QED
- arbitrariness in definition of renormalized couplings can be used to relate physical correlation functions at different energies



HELVETICA PHYSICA ACTA

VOLUMEN XXIV

MCMLI

pp 317-319

Compte rendu de la réunion de la Société Suisse de Physique

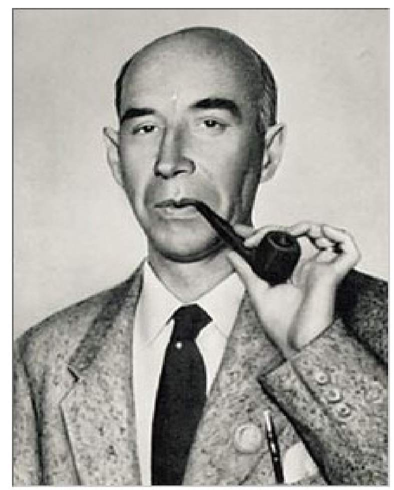
à Berne, le 5 mai 1951.

The normalization group in quantum theory

by E. C. G. Stueckelberg and A. Petermann (Genève)*).

In order to discuss complex interactions, we generalize Dyson's method[1]) in the following way:

Consider a given n-th order contribution to the S-matrix



Ernest C G Stückelberg

André Petermann

- 1951 paper remained unnoticed, even by QFT experts
- 1953 paper: finite renormalization transformations form a Lie group, for which differential equations hold. (almost unnoticed, because in French)

RG in quantum field theory: 1950s

- Gell-Mann and Low, 1954

PHYSICAL REVIEW VOLUME 95, NUMBER 5 SEPTEMBER 1, 1954

Quantum Electrodynamics at Small Distances*

M. GELL-MANN† AND F. E. LOW
Physics Department, University of Illinois, Urbana, Illinois
(Received April 1, 1954)

The renormalized propagation functions D_{FC} and S_{FC} for photons and electrons, respectively, are investigated for momenta much greater than the mass of the electron. It is found that in this region the individual terms of the perturbation series to all orders in the coupling constant take on very simple asymptotic forms. An attempt to sum the entire series is only partially successful. It is found that the series satisfy certain functional equations by virtue of the renormalizability of the theory. If photon self-energy parts are

- Bogoliubov and Shirkov, 1955

first appearance of the name renormalization group

NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE
VOL. III, N. 5 *Serie decima* 1° Maggio 1956

Charge Renormalization Group in Quantum Field Theory.

N. N. BOGOLIUBOV and D. V. SHIRKOV
Steklov Mathematical Institute of the Academy of Sciences of the USSR - Moscow

(ricevuto il 24 Novembre 1955)

Summary. — Lie differential equations are obtained for the multiplicative charge renormalization group in quantum electrodynamics and in pseudoscalar meson theory with two coupling constants. By the employment of these equations there have been found the asymptotic expressions for the electrodynamical propagation functions in the « ultraviolet » and in the « infrared » regions. The asymptotic high momenta behaviour of meson theory propagation functions is also discussed, for the weak coupling case.

Kenneth Wilson, 1970s

- problem in statistical physics: universality of critical exponents:

specific heat: $C(t) \propto |t|^{-\alpha}$,
spontaneous magnetization: $m(t) \propto (-t)^{\beta}$, $t \leq 0$,
magnetic susceptibility: $\chi(t) \propto |t|^{-\gamma}$.
critical isotherm: $m(h) \propto |h|^{1/\delta} \operatorname{sign} h$, $t = 0$

$$t = \frac{T - T_c}{T_c}$$

- new formulation of the RG idea "Wilsonian RG"

  (more general than field theoretical RG)

- Nobel Prize in Physics 1982:

  "...for his theory of critical phenomena in connection with phase transitions..."

Kenneth G. Wilson

USA

Cornell University
Ithaca, NY, USA

b. 1936

Wilsonian RG: pioneering works

- Wilson, Phys. Rev., 1971

PHYSICAL REVIEW B VOLUME 4, NUMBER 9 1 NOVEMBER 1971

Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture*

Kenneth G. Wilson
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14850
(Received 2 June 1971)

The Kadanoff theory of scaling near the critical point for an Ising ferromagnet is cast in differential form. The resulting differential equations are an example of the differential equations of the renormalization group. It is shown that the Widom-Kadanoff scaling laws arise naturally from these differential equations if the coefficients in the equations are analytic at the critical point. A generalization of the Kadanoff scaling picture involving an "irrelevant" variable is considered; in this case the scaling laws result from the renormalization-group equations only if the solution of the equations goes asymptotically to a fixed point.

- Wilson, Fisher, PRL 28, 240 (1972)

Critical Exponents in 3.99 Dimensions*

Kenneth G. Wilson and Michael E. Fisher
Laboratory of Nuclear Studies and Baker Laboratory, Cornell University, Ithaca, New York 14850
(Received 11 October 1971)

Critical exponents are calculated for dimension $d=4-\epsilon$ with ϵ small, using renormalization-group techniques. To order ϵ the exponent γ is $1+\frac{1}{6}\epsilon$ for an Ising-like model and $1+\frac{1}{5}\epsilon$ for an XY model.

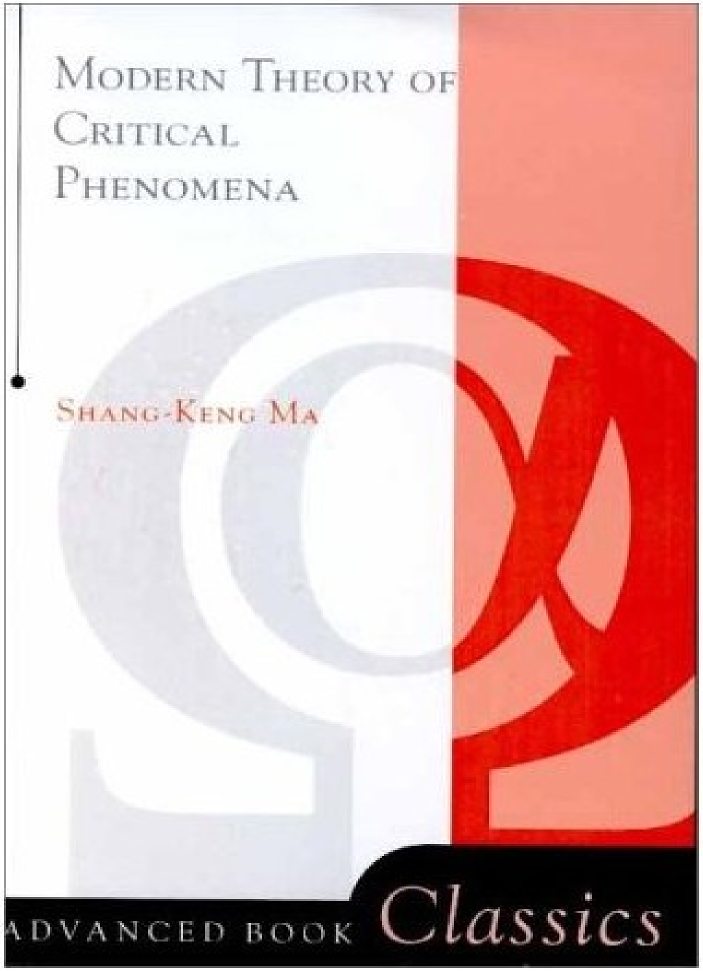
critical exponents for some universality classes

exponent	Ising_2	Ising_3	XY_3	Heisenberg_3
α	0 (log)	0.110(1)	-0.015	-0.10
β	1/8	0.3265(3)	0.35	0.36
γ	7/4	1.2372(5)	1.32	1.39
δ	15	4.789(2)	4.78	5.11
ν	1	0.6301(4)	0.67	0.70
η	1/4	0.0364(5)	0.038	0.027

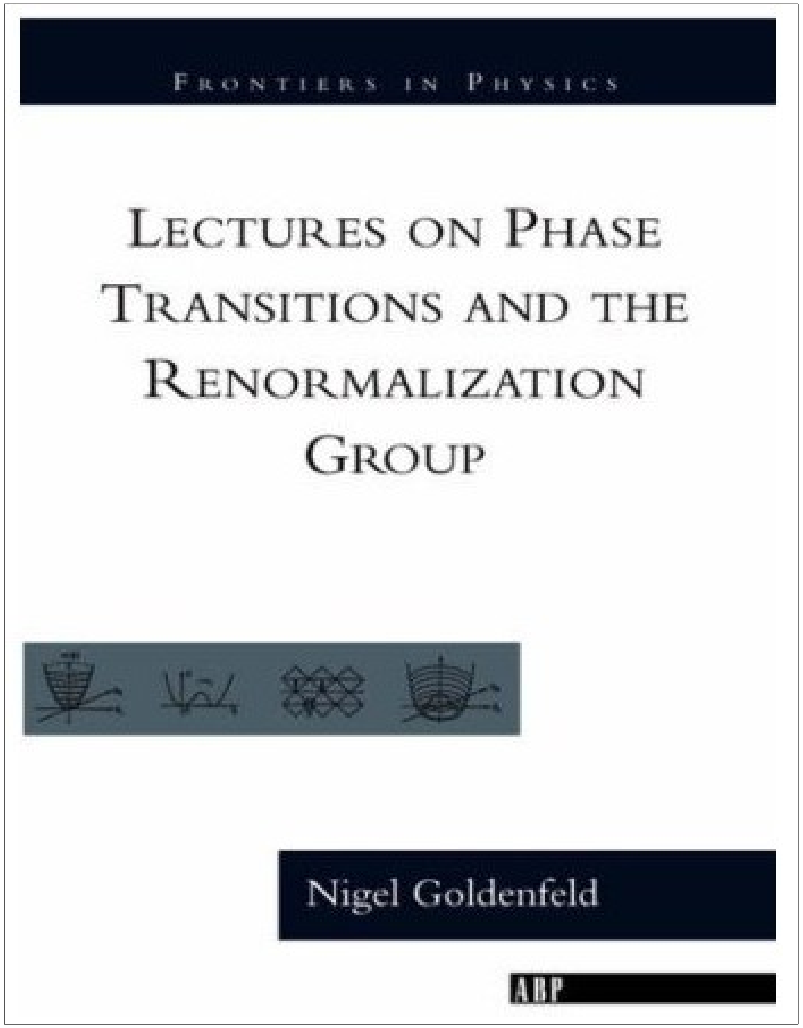
values of critical exponents cannot be obtained from dimensional analysis!

Literature: Wilsonian RG

- S. K. Ma, 1976:



- N. Goldenfeld, 1992:



Functional RG 1990s-today

- idea: derive exact functional differential equation describing Wilsonian mode elimination (Wegner and Houghton, 1972)

PHYSICAL REVIEW A VOLUME 8, NUMBER 1 JULY 1973

Renormalization Group Equation for Critical Phenomena

Franz J. Wegner
Institut für Festkörperforschung, KFA Jülich, D517 Jülich, Germany

Anthony Houghton*
Department of Physics, Brown University, Providence, Rhode Island 02912
(Received 27 October 1972)

An exact renormalization equation is derived by making an infinitesimal change in the cutoff in momentum space. From this equation the expansion for critical exponents around dimensionality 4 and the limit $n=\infty$ of the n-vector model are calculated. We obtain agreement with the results of Wilson and Fisher, and with the spherical model.

- most convenient formulation: “Wetterich equation” (Wetterich, 1993)

$$\partial_\Lambda \Gamma_\Lambda^{\mathrm{We}}[\bar\Phi] = \frac{1}{2}\mathrm{STr}\left[[\partial_\Lambda \mathbf{R}_\Lambda]\left(\mathbf{\Gamma}_\Lambda^{\mathrm{We}(2)}[\bar\Phi]+\mathbf{R}_\Lambda\right)^{-1}\right].$$

Physics Letters B 301 (1993) 90–94
North-Holland

PHYSICS LETTERS B

Exact evolution equation for the effective potential

Christof Wetterich
Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, W-6900 Heidelberg, FRG

Received 15 November 1992; revised manuscript received 17 December 1992

We derive a new exact evolution equation for the scale dependence of an effective action. The corresponding equation for the effective potential permits a useful truncation. This allows one to deal with the infrared problems of theories with massless modes in less than four dimensions which are relevant for the high temperature phase transition in particle physics or the computation of critical exponents in statistical mechanics.

Literature on FRG

Springer
the language of science

springer.com

2010. Approx. 390 p. (Lecture Notes in Physics, Vol. 798) Hardcover

- ► 69,95 €
- ► $89.95
- ► SFr. 109.00
- ► £62.99

ISBN 978-3-642-05093-0

P. Kopietz, Johann Wolfgang Goethe-Universität, Frankfurt, Germany; L. Bartosch, Johann Wolfgang Goethe-Universität,Frankfurt, Germany; F. Schütz, Johann Wolfgang Goethe-Universität, Frankfurt, Germany

Introduction to the Functional Renormalization Group

This book, based on a graduate course given by the authors, is a pedagogic and self-contained introduction to the renormalization group with special emphasis on the functional renormalization group. The functional renormalization group is a modern formulation of the Wilsonian renormalization group in terms of formally exact functional differential equations for generating functionals. In Part I the reader is introduced to the basic concepts of the renormalization group idea, requiring only basic knowledge of equilibrium statistical mechanics. More advanced methods, such as diagrammatic perturbation theory, are introduced step by step. Part II then gives a self-contained introduction to the functional renormalization group. After a careful definition of various types of generating functionals, the renormalization group flow equations for these functionals are derived. This procedure is shown to encompass the traditional method of the mode elimination steps of the Wilsonian renormalization group procedure. Then, approximate solutions of these flow equations using expansions in powers of irreducible vertices or in powers of derivatives are given. Finally, in Part III the exact hierarchy of functional renormalization group flow equations... *more on* *http://springer.com/978-3-642-05093-0*

2. Wilsonian RG: the basic idea

- explain concepts for Ising model on D-dimensional lattice, nearest neighbor coupling J, magnetic field h:

$$H = -J \sum_{\langle ij \rangle}^{N} s_i s_j - h \sum_{i=1}^{N} s_i$$

- want: partition function:

$$Z(T,h) = \sum_{\{s_i\}} e^{-\beta H} \equiv \sum_{s_1=\pm 1} \sum_{s_2=\pm 1} \cdots \sum_{s_N=\pm 1} \exp\left[\beta J \sum_{\langle ij \rangle} s_i s_j + \beta h \sum_i s_i\right]$$

- exact results only in D=1, and D=2 for h=0 (Onsager, 1944)

- first try: mean-field approximation:

$$s_i = m + \delta s_i \qquad s_i s_j = m^2 + m(\delta s_i + \delta s_j) + \cancel{\delta s_i \delta s_j}$$

$$Z_{\mathrm{MF}}(T,h) = e^{-\beta N z J m^2/2}\left[2\cosh[\beta(h + zJm)]\right]^N$$

- mean-field critical exponents wrong for D < 4

Ginzburg-Landau-Wilson action

- effective field theory for Ising model: φ^4- theory:

$$\boxed{\begin{aligned} S_{\Lambda_0}[\varphi] &= V f_0 - h_0 \varphi(\boldsymbol{k}=0) + \frac{1}{2}\int_{\boldsymbol{k}} [r_0 + c_0 \boldsymbol{k}^2] \varphi(-\boldsymbol{k})\varphi(\boldsymbol{k}) \\ &+ \frac{u_0}{4!}\int_{\boldsymbol{k}_1}\int_{\boldsymbol{k}_2}\int_{\boldsymbol{k}_3}\int_{\boldsymbol{k}_4} (2\pi)^D \delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\varphi(\boldsymbol{k}_1)\varphi(\boldsymbol{k}_2)\varphi(\boldsymbol{k}_3)\varphi(\boldsymbol{k}_4)\ , \end{aligned}}$$

- bare couplings: $f_0 = -\frac{N}{V}\ln 2 \quad h_0 = \frac{\beta^2 J_{\boldsymbol{k}=0} h}{a^{1+D/2}} \qquad c_0 = \frac{1}{z} = \frac{1}{2D}$

$$r_0 = \frac{T - T_c}{a^2 T_c} \qquad\qquad u_0 = 2a^{D-4}(\beta J_{\boldsymbol{k}=0})^4$$

- ultraviolet cutoff: $|\boldsymbol{k}| < \Lambda_0 \ll a^{-1}$
- partition function becomes functional integral: $\mathcal{Z} = \int \mathcal{D}[\varphi] e^{-S_{\Lambda_0}[\varphi]}$
- derivation: multi-dimensional Gaussian integral

$$\left(\prod_{i=1}^{N}\int_{-\infty}^{\infty}\frac{dx_i}{\sqrt{2\pi}}\right) e^{-\frac{1}{2}\boldsymbol{x}^T\mathbf{A}\boldsymbol{x}+\boldsymbol{x}^T\boldsymbol{s}} = [\det \mathbf{A}]^{-1/2} e^{\frac{1}{2}\boldsymbol{s}^T\mathbf{A}^{-1}\boldsymbol{s}}$$

Wilson's iterative RG procedure 1

- **Strategy:** perform integration iteratively in small steps:
- **Step1: Mode elimination (decimation):**

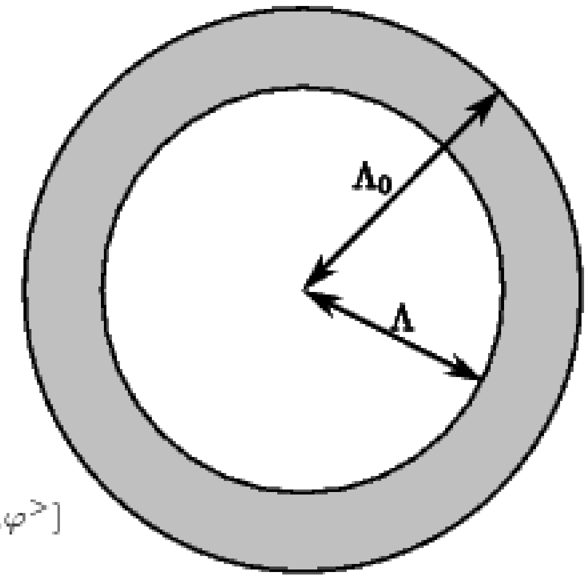
  Integrate over fields describing short wavelength or high energy fluctuations.

$$\varphi(\boldsymbol{k}) = \varphi^{<}(\boldsymbol{k}) + \varphi^{>}(\boldsymbol{k}) = \Theta(\Lambda - |\boldsymbol{k}|)\varphi(\boldsymbol{k}) + \Theta(|\boldsymbol{k}| - \Lambda)\varphi(\boldsymbol{k})$$

$$S_{\Lambda_0}[\varphi^{<} + \varphi^{>}] = S_{\Lambda}^{<}[\varphi^{<}; f_0, r_0, c_0, u_0] + S_{\Lambda,\Lambda_0}^{>}[\varphi^{>}] + S_{\mathrm{mix}}[\varphi^{<}, \varphi^{>}]$$

$$\mathcal{Z} = \int \mathcal{D}[\varphi^{<}] e^{-S_{\Lambda}^{<}[\varphi^{<}; f^{<}, r^{<}, c^{<}, u^{<}]}$$

$$e^{-S_{\Lambda}^{<}[\varphi^{<}; f^{<}, r^{<}, c^{<}, u^{<}]} = e^{-S_{\Lambda}^{<}[\varphi^{<}; f_0, r_0, c_0, u_0]} \int \mathcal{D}[\varphi^{>}] e^{-S_{\Lambda,\Lambda_0}^{>}[\varphi^{>}] - S_{\mathrm{mix}}[\varphi^{<}, \varphi^{>}]}$$

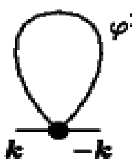


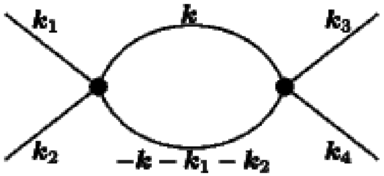
carry out integration perturbatively

Wilson's iterative RG procedure 2

- effect of mode elimination: modified couplings
  to leading (one-loop order):

$$r^{<} = r_0 + \frac{u_0}{2} \int_{\Lambda}^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{r_0 + c_0 \boldsymbol{k}^2} \qquad u^{<} = u_0 - \frac{3u_0^2}{2} \int_{\Lambda}^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{[r_0 + c_0 \boldsymbol{k}^2]^2}$$





- Step2: Rescaling: rescale wavevectors and fields such that action after mode elimination has same form as before:

$$\boldsymbol{k}' = b\boldsymbol{k} \qquad b = \Lambda_0/\Lambda$$

$$\varphi'(\boldsymbol{k}') = \zeta_b^{-1} \varphi^{<}(\boldsymbol{k}'/b)$$

$$\zeta_b = b^{1+D/2}\sqrt{Z_b}$$

$$Z_b = \frac{c_0}{c^{<}}$$

$$\rightarrow$$

$$\begin{aligned} & \frac{1}{2} \int^{\Lambda} \frac{d^D k}{(2\pi)^D} (r^{<} + c^{<} \boldsymbol{k}^2) \varphi^{<}(-\boldsymbol{k}) \varphi^{<}(\boldsymbol{k}) \\ &= \frac{1}{2} b^{-D} \int^{\Lambda_0} \frac{d^D k'}{(2\pi)^D} (r^{<} + c^{<} b^{-2} \boldsymbol{k}'^2) b^{2+D} Z_b \varphi'(-\boldsymbol{k}') \varphi'(\boldsymbol{k}') \\ &= \frac{1}{2} \int^{\Lambda_0} \frac{d^D k'}{(2\pi)^D} (b^2 Z_b r^{<} + c_0 \boldsymbol{k}'^2) \varphi'(-\boldsymbol{k}') \varphi'(\boldsymbol{k}') , \end{aligned}$$

Wilson's iterative RG procedure 3

- effect of mode elimination+rescaling:

renormalized couplings:

$$r' = b^2 Z_b r^< = b^2 Z_b \left[r_0 + \frac{u_0}{2} \int_\Lambda^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{r_0 + c_0 \boldsymbol{k}^2} \right]$$

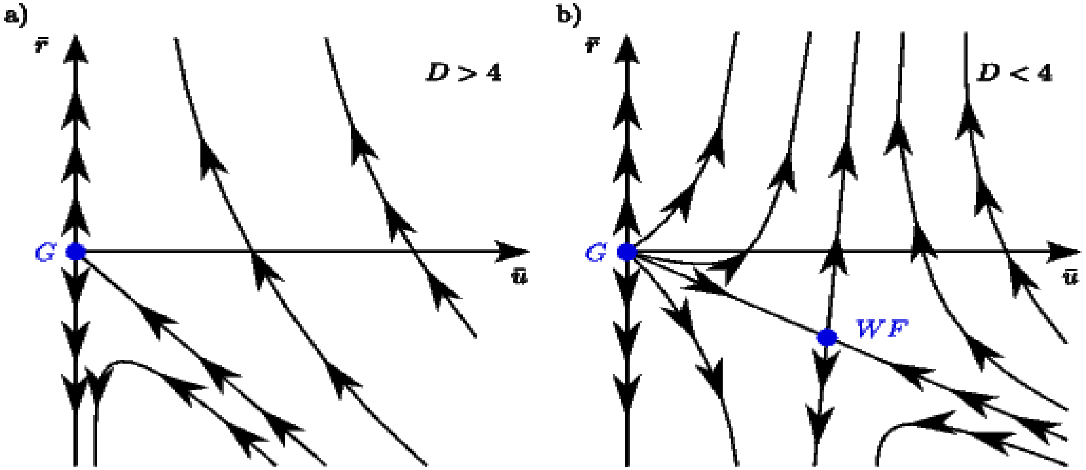
$$u' = b^{4-D} Z_b^2 u^< = b^{4-D} Z_b^2 \left[u_0 - \frac{3u_0^2}{2} \int_\Lambda^{\Lambda_0} \frac{d^D k}{(2\pi)^D} \frac{1}{[r_0 + c_0 \boldsymbol{k}^2]^2} \right]$$

- iteration in infinitesimal steps: differential equations:

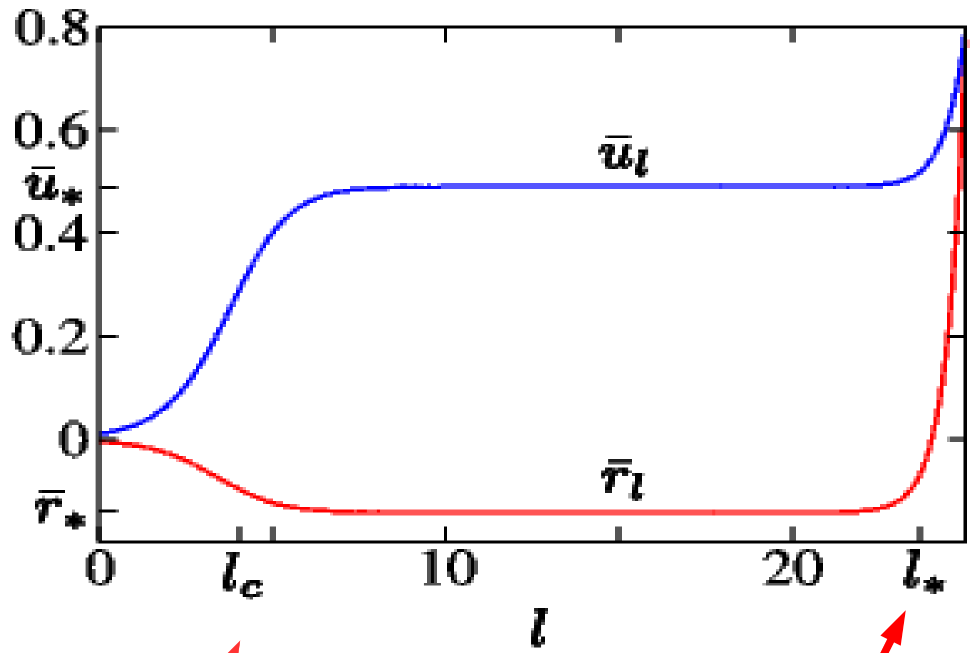
$$\Lambda = \Lambda_0 e^{-l}$$

$$\bar{r}_l = \frac{r_l}{c_0 \Lambda_0^2}, \qquad \bar{u}_l = K_D \frac{u_l}{c_0^2 \Lambda_0^{4-D}}$$

$$\boxed{\begin{aligned} \partial_l \bar{r}_l &= 2\bar{r}_l + \frac{1}{2} \frac{\bar{u}_l}{1+\bar{r}_l}, \\ \partial_l \bar{u}_l &= (4-D)\bar{u}_l - \frac{3}{2} \frac{\bar{u}_l^2}{(1+\bar{r}_l)^2}. \end{aligned}}$$



RG flow close to fixed point



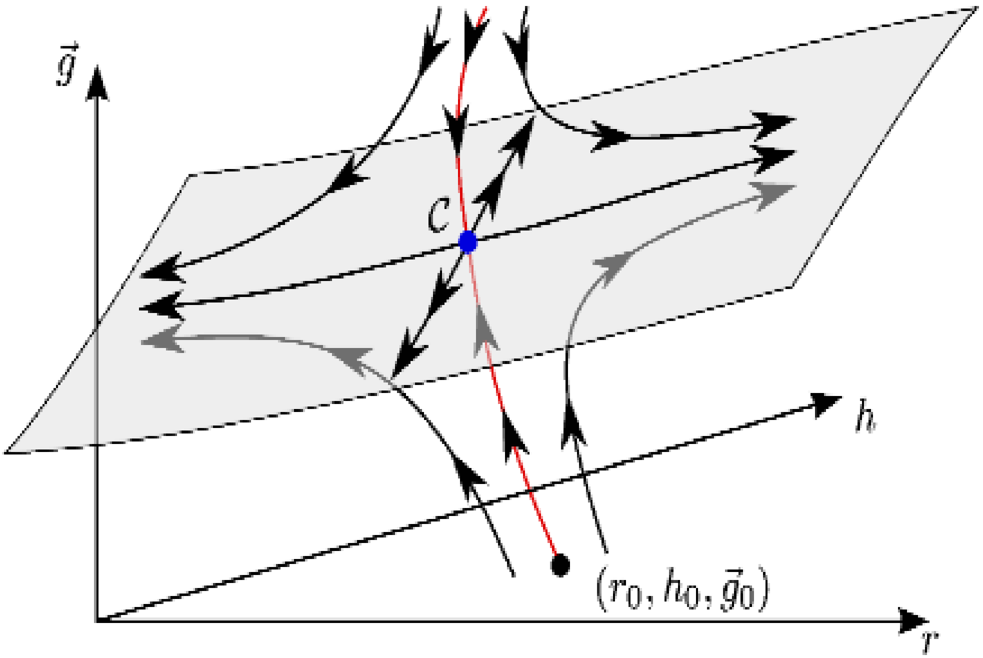
Ginzburg scale $\Lambda_0 e^{-l_c} = k_c$

inverse correlation length $\Lambda_0 e^{-l_*} = \xi^{-1}$

- RG trajectory remains for long time in vicinity of fixed point
- microscopic origin of universality

RG fixed points and critical exponents

- RG fixed points describe scale-invariant system
- critical fixed points:
  - two relevant directions
  - infinite correlation length
  - critical manifold describes system at critical point



- critical exponents:
  - are determined by eigenvalues of linearized RG flow in vicinity of critical fixed points
  - origin of universality

3. Functional renormalization group

- **main idea:** Wilsonian mode elimination can be expressed in terms of formally exact functional differential equation for generating functionals
- generating functional of Green functions

Green functions: $$G^{(n)}_{\alpha_1 \ldots \alpha_n} = \langle \Phi_{\alpha_n} \ldots \Phi_{\alpha_1} \rangle = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]} \Phi_{\alpha_n} \ldots \Phi_{\alpha_1}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}}$$

generating functional: $$\boxed{\mathcal{G}[J] = \frac{\int \mathcal{D}[\Phi] e^{-S[\Phi]+(J,\Phi)}}{\int \mathcal{D}[\Phi] e^{-S[\Phi]}} ,} \qquad G^{(n)}_{\alpha_1 \ldots \alpha_n} = \left. \frac{\delta^n \mathcal{G}[J]}{\delta J_{\alpha_n} \ldots \delta J_{\alpha_1}} \right|_{J=0}$$

example: two-point function of Ising model at critical point:

$$G^{(2)}_{\boldsymbol{r}',\boldsymbol{r}} = \langle \varphi(\boldsymbol{r}) \varphi(\boldsymbol{r}') \rangle \propto \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|^{D-2+\eta}} \qquad G^{(2)}(\boldsymbol{k}) \propto \frac{1}{|\boldsymbol{k}|^{2-\eta}}$$

$\eta \approx 0.036$ anomalous dimension in D=3.

FRG flow of generating functionals

- strategy: derive RG equation for generating functionals

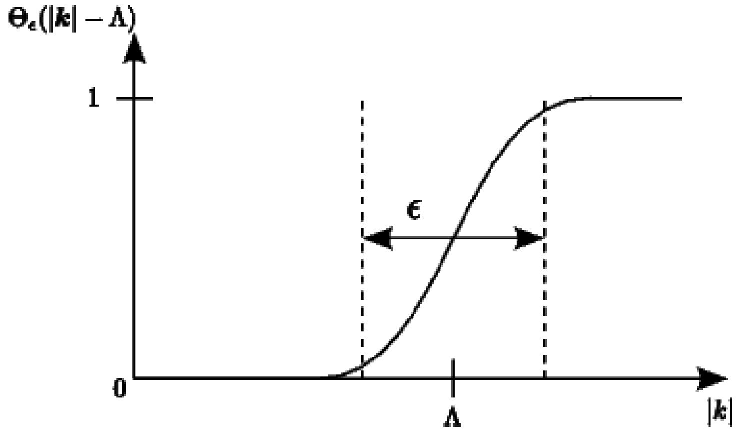
  information for RG flow of all Green functions

- introduce cutoff: modify Gaussian propagator

$$S_0[\varphi] = \frac{1}{2}\int_{\boldsymbol{k}} (r_0 + c_0 \boldsymbol{k}^2)\varphi(-\boldsymbol{k})\varphi(\boldsymbol{k}) = \frac{1}{2}\int_{\boldsymbol{k}} G_0^{-1}(\boldsymbol{k})\varphi(-\boldsymbol{k})\varphi(\boldsymbol{k}) \qquad G_0(\boldsymbol{k}) = \frac{1}{r_0 + c_0 \boldsymbol{k}^2}$$

$$G_0(\boldsymbol{k}) \to G_{0,\Lambda}(\boldsymbol{k}) = \Theta_\epsilon(|\boldsymbol{k}| - \Lambda) G_0(\boldsymbol{k})$$

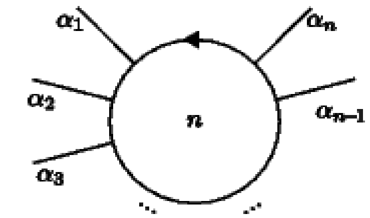
$$\Theta_\epsilon(|\boldsymbol{k}| - \Lambda) = \begin{cases} 1 & \text{for } |\boldsymbol{k}| \gg \Lambda, \\ 0 & \text{for } |\boldsymbol{k}| \ll \Lambda. \end{cases}$$



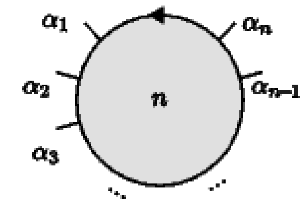
- take derivative of generating functional with respect to cutoff $\longrightarrow$ FRG flow equation

exact FRG flow equations

- different types of Green/vertex functions
  - connected Green functions $\mathcal{G}_c[J] = \ln\left(\frac{\mathcal{Z}}{\mathcal{Z}_0}\mathcal{G}[J]\right)$
    (Wegner-Houghton equation, 1972)



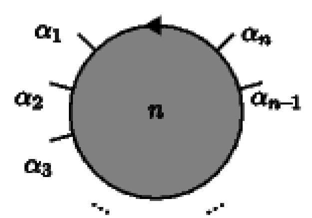
  - amputated connected Green functions
    (Polchinski equation, 1984)



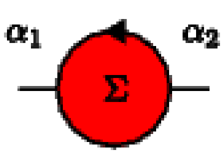
technical complication: delta-function*step function:

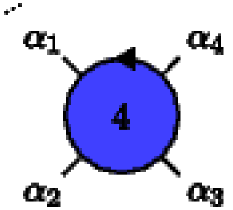
$$\delta(x)\Theta(x) = \tfrac{1}{2}\delta(x) \qquad \delta(x)\Theta^2(x) = \tfrac{1}{3}\delta(x)$$

  - one-line irreducible vertices
    (Wetterich equation, 1993)



$$\partial_\Lambda \Gamma_\Lambda^{\mathrm{We}}[\bar\Phi] = \frac{1}{2}\mathrm{Tr}\left[[\partial_\Lambda \mathbf{R}_\Lambda]\left(\frac{\delta}{\delta\bar\Phi}\otimes\frac{\delta}{\delta\bar\Phi}\Gamma_\Lambda^{\mathrm{We}}[\bar\Phi] + \mathbf{Z}\mathbf{R}_\Lambda\right)^{-1}\right]$$





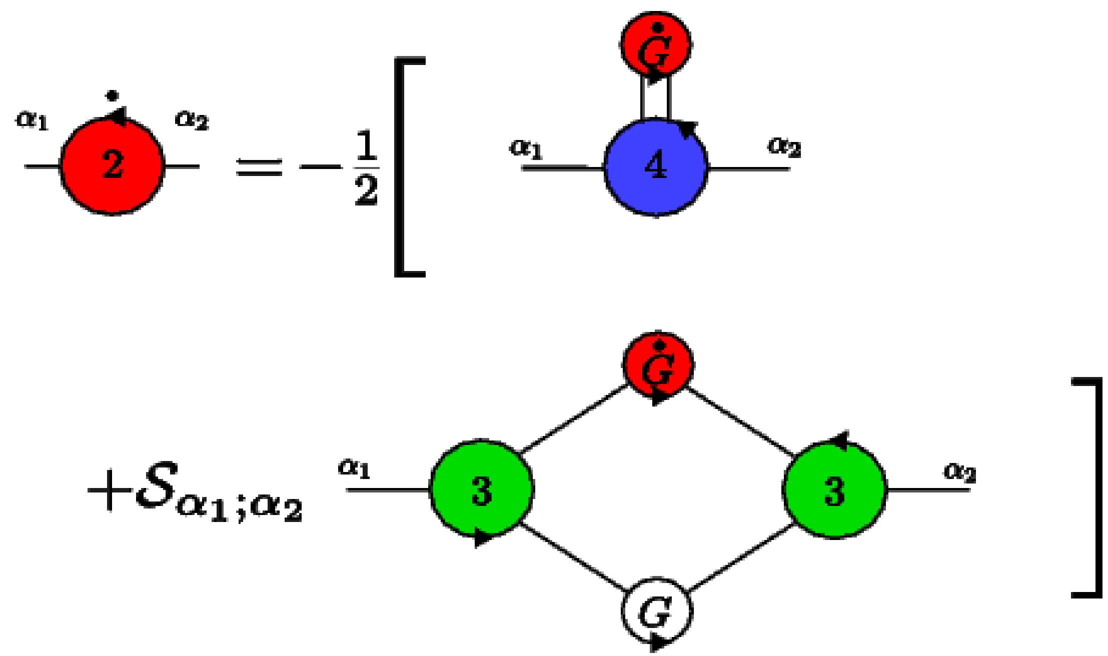
cutoff function second functional derivative

vertex expansion 1

- functional Taylor expansion $\Gamma[\bar{\Phi}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \dots \int_{\alpha_n} \Gamma^{(n)}_{\alpha_1 \dots \alpha_n} \bar{\Phi}_{\alpha_1} \dots \bar{\Phi}_{\alpha_n}$

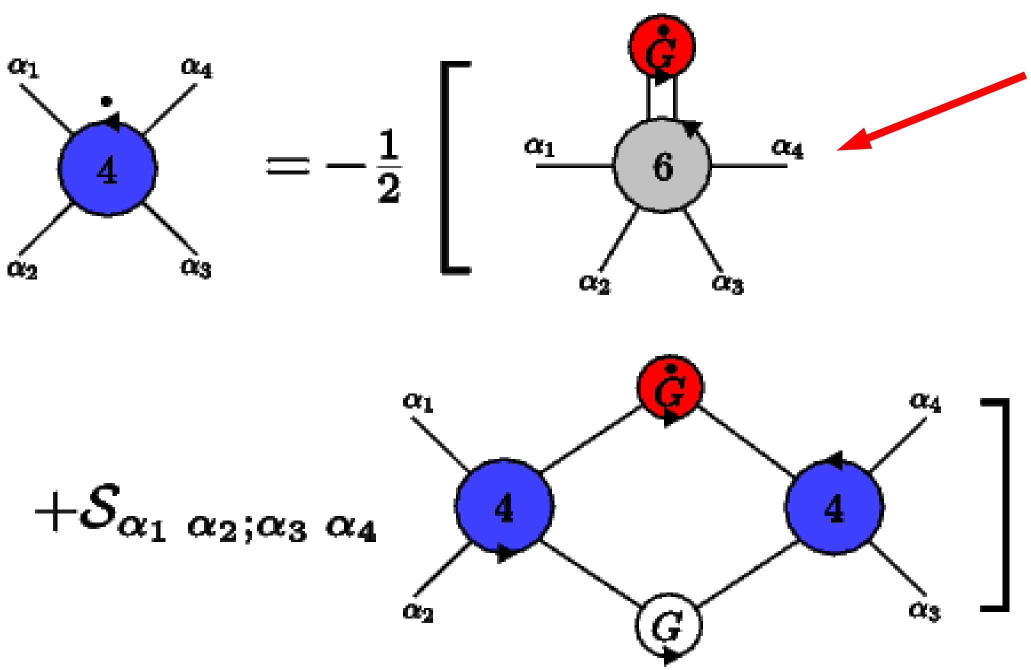
→ Wetterich equation reduces to infinite hierarchy of integro-differential equations for irreducible vertices

- exact flow equation for irreducible self-energy



vertex expansion 2

- exact flow equation for effective interaction

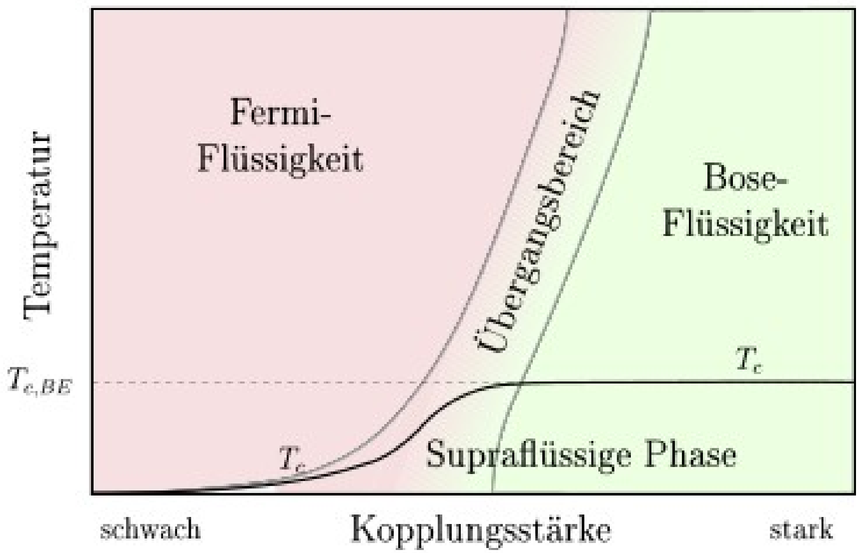


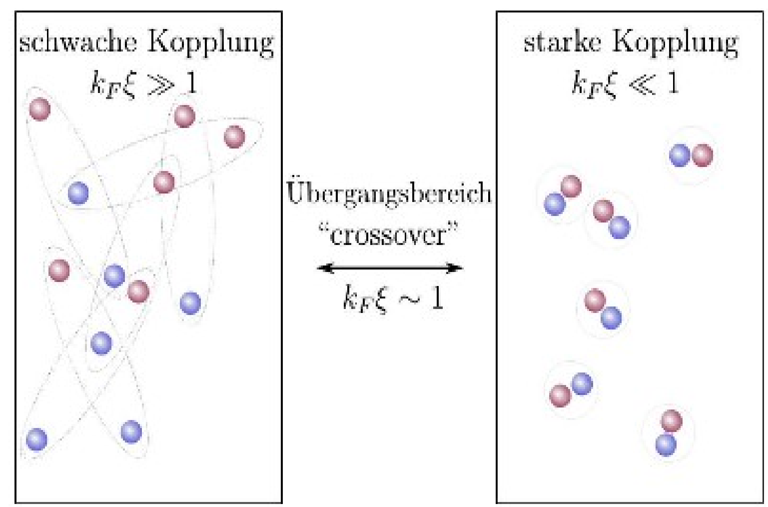
three-body renormalizes two-body interaction

- urgently needed: trunction strategies!

4. BCS-BEC crossover

- electron gas with attractive interaction:
  crossover from weakly coupled Cooper pairs
  to strongly bound fermion pairs:





- qualitative phase diagram: mean field theory ok (Eagles, 1969)
- quantitative calculations in crossover regime difficult
- experimentally accessible with ultracold atoms (also in Innsbruck)

BCS-BEC crossover: mean-field results

- model: fermions with short range two-body attraction

$$\hat{H} = \sum_{\boldsymbol{k},\sigma} \epsilon_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} - \frac{g_0}{V} \sum_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{p}} c^\dagger_{\boldsymbol{k}+\boldsymbol{p}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} c_{-\boldsymbol{k}'\downarrow} c_{\boldsymbol{k}'+\boldsymbol{p}\uparrow}$$

- regularized BCS gap equation:

$$\boxed{\frac{1}{g} = \frac{1}{V} \sum_{\boldsymbol{k}} \left[\frac{\tanh(\beta E_{\boldsymbol{k}}/2)}{2E_{\boldsymbol{k}}} - \frac{1}{2\epsilon_{\boldsymbol{k}}} \right] .}$$

$$E_{\boldsymbol{k}} = \sqrt{\xi^2_{\boldsymbol{k}} + \Delta_0^2} \qquad \xi_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} - \mu$$

$$\frac{1}{g_0} = \frac{1}{g} + \frac{1}{V} \sum_{\boldsymbol{k}} \frac{1}{2\epsilon_{\boldsymbol{k}}}$$

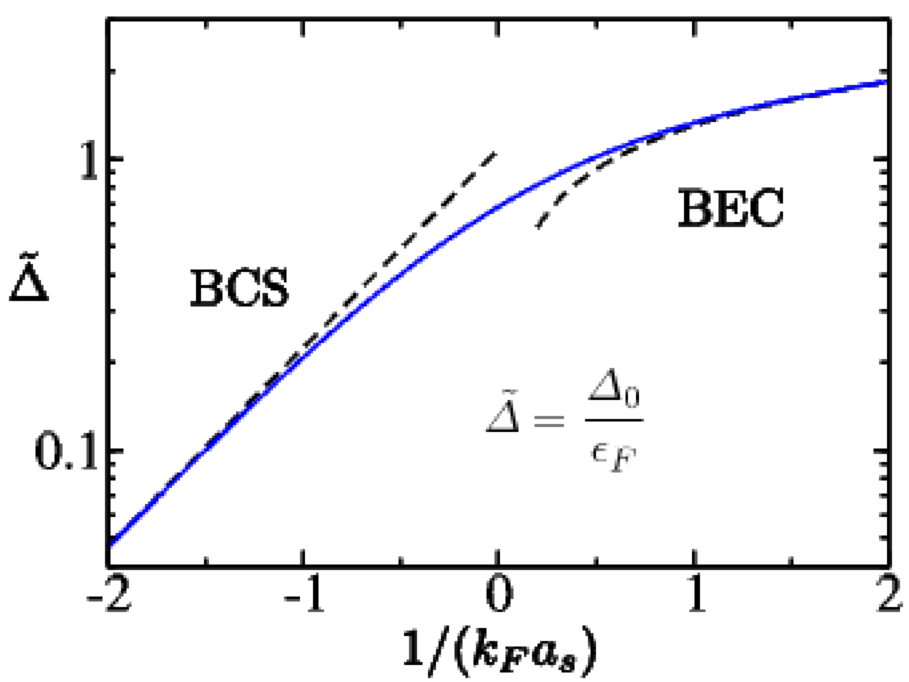
- mean-field equation for chemical potential:

$$\boxed{\rho = \frac{1}{V} \sum_{\boldsymbol{k}} \left[1 - \frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}} \tanh(\beta E_{\boldsymbol{k}}/2) \right] .}$$

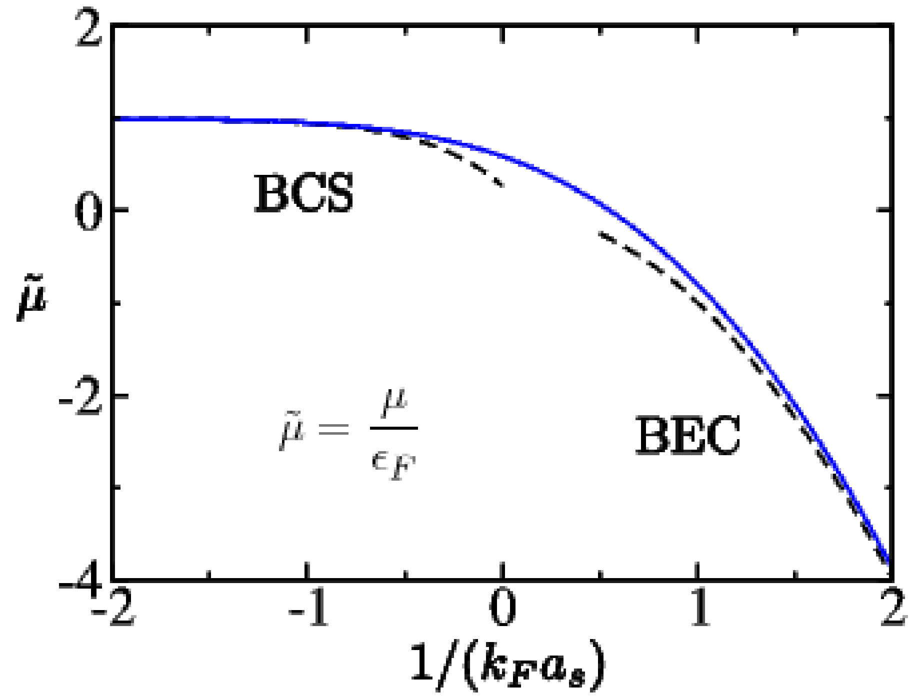
- dimensionless coupling: $\tilde{g} = \nu_0 g = -2k_F a_s/\pi$

mean-field results

- gap:



- chemical potential:

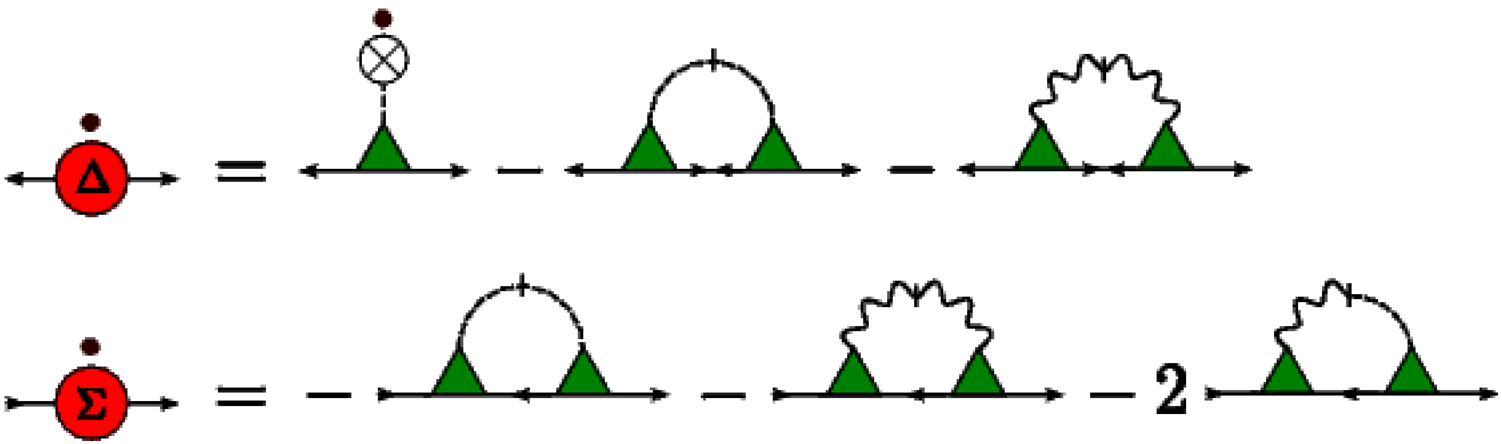


- at unitary point ($1/(k_F a_s) = 0$): $\tilde{\mu} = 0.5906$, $\tilde{\Delta} = 0.6864$

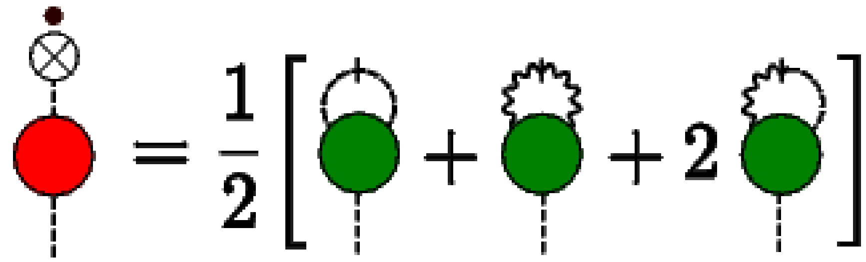
beyond mean-field: FRG flow equations

(Lorenz Bartosch, P.K., Alvaro Ferraz, PRB, 2009)

- truncated FRG flow equations for self-energies:



- flow of order parameter:



- close system of flow equations using Ward identities and skeleton equations

truncation of FRG flow equations

- Ward identity: relation between vertex functions of different order due symmetry:

$$\langle\chi\rangle \Gamma^{\bar{\psi}_\uparrow \bar{\psi}_\downarrow \chi}(K, -K; 0) = \Delta(K)$$

superfluid order parameter

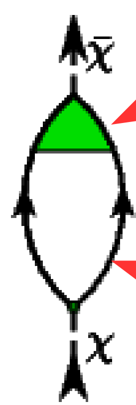
interaction vertex

single-particle gap

- skeleton equation: relation between vertex functions (not implied by symmetry):

order parameter correlation function

$$\Pi^{\bar{\chi}\chi} = -$$

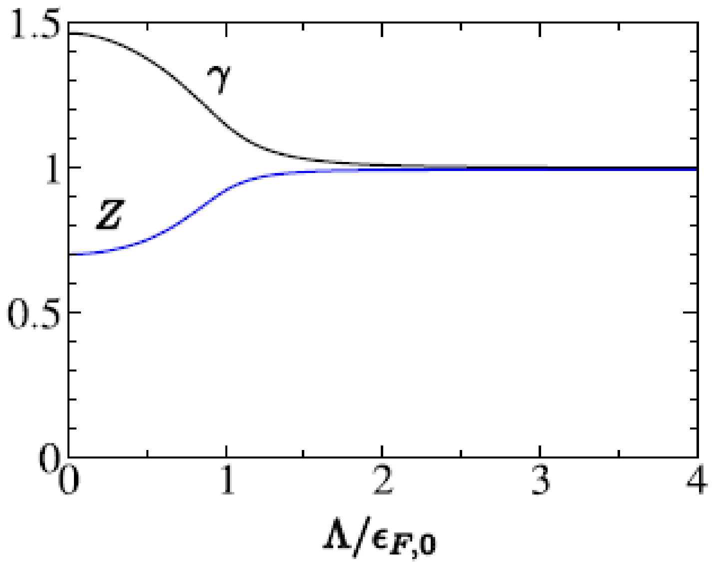


interaction vertex

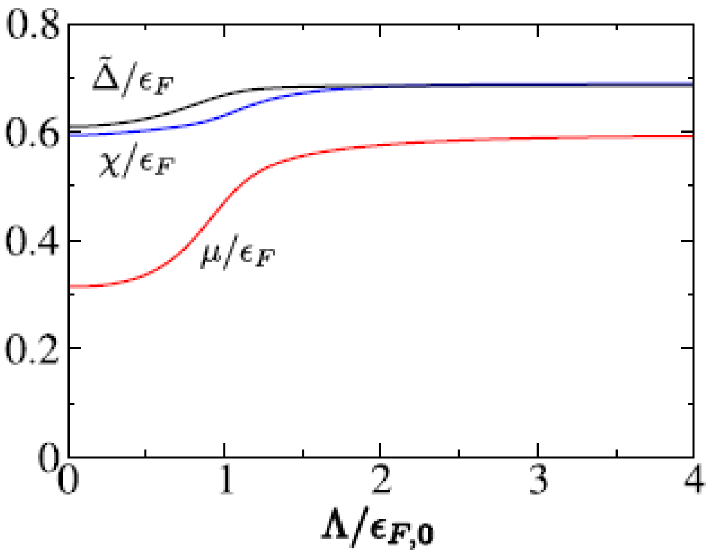
fermion propagators

FRG results:

- wave-function renormalization, vertex correction



- gap, order parameter, chemical potential



- at unitary point: $\mu/\epsilon_F = 0.32\,, \quad \tilde{\Delta}/\epsilon_F = 0.61\,, \quad \langle\chi\rangle/\epsilon_F = 0.59$

agrees with Bartenstein et al (PRL 2004)

5.Conclusions

- the renormalization group is a powerful method to analyze interacting many-body systems
- functional RG gives formally exact equations describing the Wilsonian mode elimination step
- application to BCS-BEC crossover: quantitative results, comparison with experiments
- outlook:
  - better truncation strategies needed
  - applications to systems out of equilibrium