Exact solution of the Tomonaga-Luttinger model by means of the functional renormalization group

Florian Schütz\textsuperscript{1}, Lorenz Bartosch\textsuperscript{1,2}, and Peter Kopietz\textsuperscript{2}

kopietz@itp.uni-frankfurt.de

cond-mat/0409404

\textsuperscript{1} Institut für Theoretische Physik, J.W.Goethe-Universität Frankfurt/Main

\textsuperscript{2} Department of Physics, Yale University, New Haven

Seminarvortrag Darmstadt, 10. März 2005

Outline

- Introduction
- HS transformation, derivation of functional RG equations
- Rescaling, classification of vertices
- Interaction cutoff scheme and Ward identities
- Exact solution of TL model
Introduction

- **functional renormalization group:**
  - (functional) RG for Grassmannian functional integral (Shankar ’94; Zanchi & Schulz ’96; Halboth & Metzner ’00; Honerkamp, Salmhofer, Furukawa, Rice ’01; Kopietz & Busche ’01; . . .).
  - functional integro-differential equations: a mathematical nightmare! severe truncations and hard numerical work necessary to make progress.
  - honest two-loop calculation in 2D still to be done!
• are there non-trivial examples where infinite hierarchy of coupled integro-differential equations can be solved exactly?

• **YES:** interacting electrons with dominant forward scattering!
  - solutions of infinite hierarchy of flow equations are given by infinite hierarchy of Ward identities
  - flow equation for two-point vertex can be closed and solved exactly
  - agreement with bosonization result (of course!)
  - possibility to study non-universal effects (band curvature, crossover scales etc...)

• general problem: how to calculate spectral functions of strongly correlated systems using RG methods?
Fermions in 1d

- interaction can be parameterized in terms of four marginal couplings
  - $g_2$: forward scattering, opposite Fermi points
  - $g_4$: forward scattering, same Fermi point
  - $g_1$: backward scattering
  - $g_3$: Umklapp scattering (commensurate fillings)

- RG flow in coupling space has been discussed for many years (Solyom, 1979; field theory RG).
- no RG calculations of single-particle spectral function
- Tomonaga-Luttinger model: only forward scattering: $g_2, g_4$; linear dispersion
  exactly solvable via bosonization (Luther and Peschel, 1974)
Spectral function of the TLM

- vanishing density of states at Fermi energy
- no jump in momentum distribution at Fermi surface (no quasiparticles)
- anomalous scaling
- spin-charge separation
• **Question**: is it possible to obtain the spectral properties of the TLM entirely within the framework of the RG?

• **strategy**:
  - decouple electron-electron interaction in the zero-sound channel via Hubbard-Stratonovich transformation.
  - derive exact infinite hierarchy of RG flow equations for coupled Fermi-Bose field theory
  - try so solve hierarchy exactly, guided by Ward identities

• strategy can be generalized to include other scattering channels: both zero-sound channels, particle-particle on the same footing
HS-Transformation

- aim: functional RG with collective fields (Correia, Polonyi, Richert ’01)
- start from action with only density-density interaction:

\[
S[\psi] = \sum_{\sigma} \int_{K} \bar{\psi}_{K\sigma} [-i\omega + \xi_k] \psi_{K\sigma} + \frac{1}{2} \sum_{\sigma\sigma'} \int_{\bar{K}} f^{-1}_{\bar{k}} \bar{\rho}_{\bar{K}\sigma} \rho_{\bar{K}\sigma'} \quad \rho_{\bar{K}\sigma} = \int_{K} \bar{\psi}_{K\sigma} \psi_{K+\bar{K},\sigma}
\]

Hubbard-Stratonovich transformation

\[
S[\psi, \varphi] = \sum_{\sigma} \int_{K} \bar{\psi}_{K\sigma} [-i\omega + \xi_k] \psi_{K\sigma} + \frac{1}{2} \sum_{\sigma\sigma'} \int_{\bar{K}} [f^{-1}_{\bar{k}}] \bar{\phi}^*_{\bar{K}\sigma} \varphi_{\bar{K}\sigma'} + i \sum_{\sigma} \int_{\bar{K}} \bar{\rho}_{\bar{K}\sigma} \varphi_{\bar{K}\sigma}.
\]

\[\rightarrow\]
Efficient notation

- collect fields in vector:
  \[ \Phi = (\psi, \bar{\psi}, \varphi) \]

- symmetrized quadratic part of action:
  \[ S_0[\Phi] = -\frac{1}{2} \left( \Phi, [G_0]^{-1} \Phi \right) \]

- generalized free propagator (\( \zeta = \pm 1 \) for bosons/fermions):
  \[ G_0^{-1} = \begin{pmatrix}
  0 & \zeta[G_0^{-1}]^T & 0 \\
  G_0^{-1} & 0 & 0 \\
  0 & 0 & -F_0^{-1}
\end{pmatrix}, \quad G_0^T = ZG_0, \quad Z = \begin{pmatrix}
  \zeta & 0 & 0 \\
  0 & \zeta & 0 \\
  0 & 0 & 1
\end{pmatrix} \]

- with bare Green function and bare interaction:
  \[
  [G_0]_{K\sigma, K'\sigma'} = \delta_{K, K'} \delta_{\sigma \sigma'} G_{0, \sigma}(K) \quad G_{0, \sigma}(K) = [i\omega - \xi_{k\sigma}]^{-1}, \\
  [F_0]_{K\sigma, K'\sigma'} = \delta_{K, -K'} F_{0, \sigma \sigma'}(\bar{K}) \quad F_{0, \sigma \sigma'}(\bar{K}) = f_{\bar{K}}^{\sigma \sigma'}
  \]
Generating functionals

- generating functional for Green functions:

\[ G[J] = e^{G_c[J]} = \frac{1}{Z_0} \int D\Phi \, e^{-S_0 - S_{\text{int}} + (J, \Phi)}, \]

- with:

\[ (J, \Phi) = (\tilde{J}, \psi) + (\bar{\psi}, j) + (J^*, \phi). \]

- Legendre transformation:

\[ \Phi := \frac{\delta G_c}{\delta J}, \quad \mathcal{L}[\Phi] := (J[\Phi], \Phi) - G_c[J[\Phi]], \quad \Rightarrow \quad J = Z \frac{\delta \mathcal{L}}{\delta \Phi}. \]

- effective action generates one-line-irreducible (1LI) vertices:

\[ \Gamma[\Phi] := \mathcal{L}[\Phi] - S_0[\Phi]. \]
Tree expansion

- relation between connected and 1LI vertices:

\[ 1 = \frac{\delta \Phi}{\delta \Phi} = \frac{\delta^2 L}{\delta \Phi \delta \Phi} Z \frac{\delta^2 G_c}{\delta J \delta J} . \]

- higher derivatives:
Cutoffs

- **band cutoff:**

\[ G_0(K) \rightarrow \theta(\Lambda < D_K < \Lambda_0) G_0(K) \]
\[ D_K = \frac{|\epsilon_k - \epsilon_{k_F}|}{v_0} \]

- **interaction cutoff:**

\[ F_0(\bar{K}) \rightarrow \theta(\Lambda < \bar{D}_{\bar{K}} < \Lambda_0) F_0(\bar{K}) \]
\[ \bar{D}_{\bar{K}} = |\bar{k}| \]

- or: only one of these

\[ \Rightarrow \text{new RG schemes due to interaction cutoff} \]
Functional RG equations

- functional RG equation:

\[
\partial_\Lambda \Gamma = -\frac{1}{2} \text{Tr} \left[ Z \dot{G}^T U^T \left\{ 1 - G^T U^T \right\}^{-1} \right] \\
- \frac{1}{2} \text{Tr} \left[ Z \dot{G}_0^T \Sigma^T \left\{ 1 - G^T \Sigma^T \right\}^{-1} \right].
\]

- with

\[
U^T := \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} - \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} \bigg|_{\Phi=0} = \frac{\delta^2 \Gamma}{\delta \Phi \delta \Phi} - \Sigma^T,
\]

- and single scale propagator:

\[
\dot{G} = -G \partial_\Lambda [G_0^{-1}] G.
\]
Flow of irreducible vertices

- expand $\Gamma$ in powers of $\Phi$
- diagrammatics:

\[
\begin{align*}
\begin{array}{c}
2 & \rightarrow & 2 & = & - \frac{1}{2} \\
3 & \rightarrow & 3 & = & - \frac{1}{2}
\end{array}
\end{align*}
\]

\[
\begin{align*}
&\left[\begin{array}{c}
2 & \rightarrow & 4 & = & + S_{1;2} \\
3 & \rightarrow & 5 & = & + S_{12;3} \\
4 & \rightarrow & 3 & = & + S_{1;23} \\
5 & \rightarrow & 4 & = & + S_{1;2;3}
\end{array}\right]
\end{align*}
\]
RG eqs. for physical correlation functions

- use unsymmetrized vertices and usual propagators
- pictorial dictionary:

\[ \begin{align*}
2n,m & \overset{2n+m}{\longrightarrow} n', \ldots, n, I, m, I' \quad \text{flow equation for irreducible polarization:} \\
\begin{array}{c}
\begin{tikzpicture}
  \node[shape=circle,draw] (n) at (0,0) {0,2};
  \node[shape=circle,draw] (m) at (1,0) {2,2};
  \node[shape=circle,draw] (p) at (2,0) {4,0};
  \node[shape=circle,draw] (q) at (3,0) {2,2};
  \node[shape=circle,draw] (r) at (4,0) {0,4};
  \draw[->,thick] (n) -- (m);
  \draw[->,thick] (m) -- (p);
  \draw[->,thick] (p) -- (q);
  \draw[->,thick] (q) -- (r);
\end{tikzpicture}
\end{array}
\end{align*} \]

\[ \begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node[shape=circle,draw] (n) at (0,0) {2,1};
  \node[shape=circle,draw] (m) at (1,0) {2,1};
  \node[shape=circle,draw] (p) at (2,0) {2,1};
  \node[shape=circle,draw] (q) at (3,0) {2,1};
  \node[shape=circle,draw] (r) at (4,0) {2,1};
  \draw[->,thick] (n) -- (m);
  \draw[->,thick] (m) -- (p);
  \draw[->,thick] (p) -- (q);
  \draw[->,thick] (q) -- (r);
\end{tikzpicture}
\end{array}
\end{align*} \]

\[ \begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node[shape=circle,draw] (n) at (0,0) {2,0};
  \node[shape=circle,draw] (m) at (1,0) {4,0};
  \node[shape=circle,draw] (p) at (2,0) {2,2};
  \node[shape=circle,draw] (q) at (3,0) {2,2};
  \node[shape=circle,draw] (r) at (4,0) {2,2};
  \draw[->,thick] (n) -- (m);
  \draw[->,thick] (m) -- (p);
  \draw[->,thick] (p) -- (q);
  \draw[->,thick] (q) -- (r);
\end{tikzpicture}
\end{array}
\end{align*} \]

- flow equation for self-energy:

\[ \begin{align*}
\begin{array}{c}
\begin{tikzpicture}
  \node[shape=circle,draw] (n) at (0,0) {2,1};
  \node[shape=circle,draw] (m) at (1,0) {2,1};
  \node[shape=circle,draw] (p) at (2,0) {2,1};
  \node[shape=circle,draw] (q) at (3,0) {2,1};
  \node[shape=circle,draw] (r) at (4,0) {2,1};
  \draw[->,thick] (n) -- (m);
  \draw[->,thick] (m) -- (p);
  \draw[->,thick] (p) -- (q);
  \draw[->,thick] (q) -- (r);
\end{tikzpicture}
\end{array}
\end{align*} \]
vertex corrections:
Rescaling, classification of vertices

- dimensionless bosonic momenta and frequencies:
  \[ \bar{q} = \bar{k}/\Lambda , \quad \bar{\epsilon} = \bar{\omega}/\bar{\Omega}_\Lambda , \quad \bar{\Omega}_\Lambda \propto \Lambda^{\bar{z}_\phi} . \]

- for fermionic momenta use patching construction:
  \[ q = (k - k_{F,\sigma})/\Lambda , \quad \epsilon = \omega/\Omega_\Lambda , \quad \Omega_\Lambda \propto \Lambda^{z_{\psi}} . \]

- rescaling of fields, including anomalous rescaling:
  \[ \psi_{K\sigma} = \left( \frac{Z}{\Lambda^D \Omega_\Lambda^2} \right)^{1/2} \tilde{\psi}_{Q\sigma} , \quad \phi_{\bar{K}\sigma} = \left( \frac{\bar{Z}}{\Lambda^D \bar{\Omega}_\Lambda \nu_0} \right)^{1/2} \tilde{\phi}_{\bar{Q}\sigma} . \]

- anomalous dimensions, \( \Lambda = \Lambda_0 e^{-l} \):
  \[ \eta_l = -\partial_l \ln Z , \quad \bar{\eta}_l = -\partial_l \ln \bar{Z} . \]
rescaled vertices with flow equations:

\[ \partial_l \tilde{\Gamma}^{(2n,m)}_l = \left[ D^{(2n,m)} - n \eta_l - \frac{m}{2} \tilde{\eta}_l - \sum_{i=1}^{n} (Q'_i \frac{\partial}{\partial Q'_i} + Q_i \frac{\partial}{\partial Q_i}) - \sum_{i=1}^{m} \bar{Q}_i \frac{\partial}{\partial \bar{Q}_i} \right] \tilde{\Gamma}^{(2n,m)}_l \]

\[ + \tilde{\Gamma}^{(2n,m)}_l, \]

with:

\[ Q \frac{\partial}{\partial Q} \equiv \mathbf{q} \cdot \nabla \mathbf{q} + z_\psi \varepsilon \frac{\partial}{\partial \varepsilon}, \quad \bar{Q} \frac{\partial}{\partial Q} \equiv \bar{\mathbf{q}} \cdot \nabla \bar{\mathbf{q}} + z_\phi \bar{\varepsilon} \frac{\partial}{\partial \bar{\varepsilon}}. \]

scaling dimension of vertices:

\[ D^{(2n,m)} = \begin{cases} 
(1-n)D + z_{\text{min}} - (D + z_\varphi)m/2 & \text{for } n \geq 1 \\
(D + z_\varphi)(1-m/2) & \text{for } n = 0
\end{cases} \]

for Tomonaga-Luttinger model:

\[ D = 1, \ z_\psi = z_\phi = 1 \quad \Rightarrow \quad D^{(2n,m)} = 2 - n - m \]
Schwinger Dyson equation, skeleton expansion

- infinitesimal shifts in the integration variables $\Phi_\alpha$, Schwinger-Dyson equation:

$$\left( \zeta_\alpha J_\alpha - \frac{\delta S}{\delta \Phi_\alpha} \left[ \frac{\delta}{\delta J_\alpha} \right] \right) G[J_\alpha] = 0$$

- translate to equation for $\Gamma$ and expand in powers of fields:

\[
\begin{align*}
\begin{array}{ccc}
2,0 & = & 2,1 \\
0,2 & = & -\zeta \\
2,1 & = & -\zeta
\end{array}
\end{align*}
\]

\[
\begin{array}{ccc}
4,0 & = & i
\end{array}
\]

\[
\begin{array}{ccc}
i & = & -\zeta
\end{array}
\]
A simple truncation scheme

- keep only leading skeleton elements:

- numerical solution?
Interaction cutoff scheme

- from now on: only interaction cutoff, exact flow equations:

- hierarchy with respect to no. of Fermi lines
  # on r.h.s of flow equation ≤ # on l.h.s.
Initial condition

• Fermi loops

\[ 0, m = \sum_P P(1) P(2) \ldots P(m) \]

• bare vertex:

\[ \sim i \]

• all other vertices vanish

• linear dispersion: closed-loop theorem

\[ \Gamma^{(0,m)}|_{\Lambda=\Lambda_0} = 0, \quad m > 2. \]

⇒ pure boson vertices don’t flow

\[ \partial_\Lambda \Gamma^{(0,m)} = 0. \]
Flow without higher boson vertices

\[ \begin{array}{c}
2,0 \quad = \quad \frac{1}{2} \quad 2,2 \\
\end{array} \]

attach an additional Boson leg in all possible ways

\[ \begin{array}{c}
2,1 \\
\end{array} \]

simple structure, solve complete hierarchy?
Ward identities

- Action in real space and imaginary time \((X = (\tau, r))\):

\[
S[\bar{\psi}, \psi, \varphi] = S_0[\bar{\psi}, \psi] + S_0[\varphi] + S_1[\bar{\psi}, \psi, \varphi]
\]

\[
S_0[\bar{\psi}, \psi] = \sum_{\sigma} \int_X \bar{\psi}_{\sigma}(X) \partial_\tau \psi_{\sigma}(X) + \sum_{\sigma} \int d\tau \int d^D r d^D r' \bar{\psi}_{\sigma}(\tau, r) \xi_{\sigma}(r-r') \psi_{\sigma}(\tau, r')
\]

\[
S_1[\bar{\psi}, \psi, \varphi] = i \sum_{\sigma} \int_X \bar{\psi}_{\sigma}(X) \psi_{\sigma}(X) \varphi_{\sigma}(X)
\]

- Local gauge transformation: \(\psi_{\sigma}(X) = e^{i\alpha_{\sigma}(X)} \tilde{\psi}_{\sigma}\)

- Expand generating functional to first order in \(\alpha\):

\[
0 = \int_K \left\{ [i\tilde{\omega} - \xi_{k + \tilde{k}, \sigma} + \xi_{k \sigma}] \frac{\delta^2 G}{\delta j_{K\sigma} \delta j_{K + \tilde{k}\sigma}} + \tilde{j}_{K + \tilde{k}\sigma} \frac{\delta G}{\delta j_{K\sigma}} - j_{K\sigma} \frac{\delta G}{\delta j_{K + \tilde{k}\sigma}} \right\}
\]

- Linearize dispersion: \(\xi_{k + \tilde{k}, \sigma} - \xi_{k \sigma} \rightarrow v_{F, \sigma} \cdot \tilde{k}\)

use Dyson-Schwinger equation, master Ward identity:

\[
0 = (i\tilde{\omega} - v_{F, \sigma} \cdot \tilde{k}) \left[ \frac{\delta \Gamma}{\delta \varphi_{K\sigma}} - i \int_K \bar{\psi}_{K + \tilde{k}\sigma} \psi_{K\sigma} \right] + i \int_K \left[ \psi_{K\sigma} \frac{\delta \Gamma}{\delta \psi_{K + \tilde{k}\sigma}} - \bar{\psi}_{K + \tilde{k}\sigma} \frac{\delta \Gamma}{\delta \bar{\psi}_{K\sigma}} \right]
\]
WIs as solution of flow equations

- Ward identity for vertex correction:

\[
G(K + \tilde{K}) \Gamma^{(2,1)}(K + \tilde{K}; K; \tilde{K}) G(K) = \frac{-i}{i\omega - v_{F,\sigma} \cdot \hat{k}} \left[ G(K + \tilde{K}) - G(K) \right]
\]

- Diagrammatically:

consistent with flow equations:

- ids trivially fulfilled initially
- insert into flow for \( \Gamma^{(2,m+1)} \)
- intermediate 'breaks' cancel out
- difference of flow of \( \Gamma^{(2,m)} \) remains
- ids are conserved

\( \Rightarrow \) WIs are valid at every stage of RG flow
Exact solution of TL model

- Ward ids in flow $\Rightarrow$ closed flow equation for self-energy:

$$\partial_\Lambda \Sigma_\sigma(K) = G_\sigma^{-2}(K) \int_{\bar{K}} \frac{\dot{F}_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \vec{v}_{F,\sigma} \cdot \vec{K})^2} \left[ G_\sigma(K) - G_\sigma(K + \bar{K}) \right]$$

- Linear integro-differential equation for Green function:

$$\partial_\Lambda G_\sigma(K) = \int_{\bar{K}} \frac{\dot{F}_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \vec{v}_{F,\sigma} \cdot \vec{K})^2} \left[ G_\sigma(K) - G_\sigma(K + \bar{K}) \right] =: H_\Lambda(\bar{K})$$

- Solve by Fourier transformation:

$$G_\sigma(i\omega, k) = \int dx \int d\tau G_\sigma(\tau, x) e^{i\omega \tau - ikx}$$
- flow equation in real space:

\[
\left[ \partial_\Lambda + H_\Lambda(X) - H_\Lambda(0) \right] G_\sigma(X) = 0,
\]

- solution as in bosonization:

\[
G_\sigma(X) = G_{\sigma,0}(X) e^{Q_\sigma(X)}
\]

- with Debye-Waller factor

\[
Q_\sigma(X) = -S_\sigma(X) + S_\sigma(0)
\]

- and

\[
S_\sigma(X) = -\int_0^{\Lambda_0} d\Lambda' H_{\Lambda'}(X) = \int_{\tilde{K}} \frac{\theta(|\tilde{k}| < \Lambda_0) F_{\sigma\sigma}(\tilde{K})}{(i\tilde{\omega} - v_{F,\sigma} \cdot \tilde{k})^2} \cos(\tilde{\omega}\tau - \tilde{k} \cdot x)
\]
Truncation schemes

- for TL-model $\eta_t^\phi = 0$
- simplest approximation: keep only relevant and marginal terms on r.h.s. of rescaled flow equation
- coupling constants:

$$
\tilde{\mu} = \tilde{G}^{-1} \bigg|_{Q=0}, \quad \tilde{v} = \partial_q \tilde{G}^{-1} \bigg|_{Q=0}, \quad \tilde{\lambda} = \tilde{\Gamma}^{(2,1)}(0).
$$

- $\tilde{\mu}$ has to be fine tuned, $\tilde{v}$ and $\tilde{\lambda}$ don’t flow.
- anomalous dimension as in exact solution:

$$
\eta = -\partial_i \tilde{\Sigma} \bigg|_{Q=0} = \frac{\tilde{g}_0^2}{2\sqrt{1+\tilde{g}_0}[\sqrt{1+\tilde{g}_0}+1]^2}, \quad \tilde{g}_0 = \frac{g_0}{2\pi v_F}.
$$

- integrate to obtain physical self-energy:

$$
\Sigma_\alpha(K) = -\int^{+\infty}_{-\infty} \frac{d\bar{\omega}}{2\pi} \int^{k_c}_{-k_c} \frac{d\bar{k}}{2\pi} \frac{F_{\alpha\alpha}(i\bar{\omega},\bar{k})}{i(\omega + \bar{\omega}) - \alpha v_F(k + \bar{k}) + \tilde{\mu} v_F \bar{k}} \left( \frac{k_c}{|\bar{k}|} \right)^\eta.
$$
Conclusions

Summary:

- Introduce collective variables in fRG from the very beginning
- new RG schemes due to:
  - 1LI vertices
  - momentum transfer cutoff in the interaction
- exact solution of TL model is recovered, ingredients:
  - closed loop theorem as initial condition
  - Ward identities to close flow equations

Outlook:

- truncation schemes when Ward ids are not valid
- decoupling in other channels (work in progress)
- broken symmetry
- renormalization of the Fermi surface (S.Ledowski, A. Ferraz, P.K, cond-mat/0412620)