Renormalization group approach to interacting fermions

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Outline

- Introduction
- Functional RG equations
- Tomonaga-Luttinger model
- Fermi surface problem
- Symmetry breaking (in progress...)

1. Introduction

- functional renormalization group:
  - Wegner, Houghton, 1973: infinite hierarchy of coupled integro-differential equations for momentum-dependent vertex functions
  - (functional) RG for Grassmannian functional integral (Shankar ’94; Zanchi & Schulz ’96; Halboth & Metzner ’00; Honerkamp, Salmhofer, Furukawa, Rice ’01; Kopietz & Busche ’01; . . .).
  - functional integro-differential equations: a mathematical nightmare! severe truncations and hard numerical work necessary to make progress.
  - honest two-loop calculation in 2D still to be done!

1...super-field notation

- general field theory with multi-component field Φ_α
- α = super label: frequency, momentum, spin, field-type (Boson or Fermion, Majorana fermion), flavour...
- starting point: general Euclidean action

$$S[\Phi] = S_0[\Phi] + S_{\text{int}}[\Phi]$$

with (anti) symmetrized quadratic part:

$$S_0[\Phi] = -\frac{1}{2}\left(\Phi, [\mathbf{G}_0]^{-1}\Phi\right) = -\frac{1}{2}\int_\alpha\int_{\alpha'} \Phi_\alpha \left[\mathbf{G}_0\right]^{-1}_{\alpha\alpha'} \Phi_{\alpha'}$$

- for spinless fermions: $\Phi = (\psi, \overline{\psi})$, statistics factor $\zeta = -1$

$$\mathbf{G}_0^{-1} = \begin{pmatrix} 0 & \zeta[G_0^{-1}]^T \\ G_0^{-1} & 0 \end{pmatrix}, \qquad \begin{aligned} \mathbf{G}_0^T &= \zeta \mathbf{G}_0 \\ [G_0]^{-1}_{K,K'} &= \delta_{K,K'}[i\omega - \xi_k]^{-1} \end{aligned}$$

1...generating functionals

notation: F. Schütz, L. Bartosch, P.K., PRB 2005

- generating functional for Green functions:

$$\mathcal{G}[J] = e^{\mathcal{G}_c[J]} = \frac{1}{Z_0}\int D\Phi\, e^{-S_0 - S_{\rm int} + (J,\Phi)},$$

- Legendre transformation:

$$\Phi := \frac{\delta \mathcal{G}_c}{\delta J}, \qquad \mathcal{L}[\Phi] := (J[\Phi],\Phi) - \mathcal{G}_c[J[\Phi]], \quad \Rightarrow \quad J = \mathbf{Z}\frac{\delta \mathcal{L}}{\delta \Phi}.$$

$\mathbf{Z}$ = statistics matrix: $Z_{\alpha\alpha'} = \delta_{\alpha\alpha'}\zeta_\alpha$, $\zeta_\alpha = \pm 1$ for bosons/fermions

- generating functional of one-line-irreducible vertices:

$$\begin{aligned}\Gamma[\Phi] &= \mathcal{L}[\Phi] - S_0[\Phi] \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{\alpha_1}\cdots\int_{\alpha_n}\Gamma^{(n)}_{\alpha_1\ldots\alpha_n}\Phi_{\alpha_1}\ldots\Phi_{\alpha_n}.\end{aligned}$$

2. Functional RG: cutoffs

- RG idea: partial integration: introduce cutoff in free propagator
- for fermions: band cutoff

$$\begin{aligned} G_0(K) &\longrightarrow \theta(\Lambda < D_K < \Lambda_0)\, G_0(K) \\ D_K &= |\varepsilon_k - \varepsilon_{k_F}|/v_0 \end{aligned}$$

- for bosons: momentum cutoff:

$$\begin{aligned} F_0(\bar{K}) &\longrightarrow \theta(\Lambda < \bar{D}_{\bar{K}} < \Lambda_0)\, F_0(\bar{K}) \\ \bar{D}_{\bar{K}} &= |\bar{k}| \end{aligned}$$

if boson mediates effective interaction in Fermi systems (phonon):
Λ = momentum transfer cutoff

- for mixed fermi/bose problems: combination or only one of these

$\Rightarrow$ new RG schemes due to interaction cutoff $\Leftarrow$

2... exact RG equation for one-line IR vertices

Simple derivation of functional RG equation for $\Gamma[\Phi]$:
take derivative with respect to the cutoff:

$$\boxed{\begin{aligned}\partial_\Lambda \Gamma \;=\; & -\tfrac{1}{2}\mathrm{Tr}\left[\mathbf{Z}\dot{\mathbf{G}}^T\mathbf{U}^T\left\{\mathbf{1}-\mathbf{G}^T\mathbf{U}^T\right\}^{-1}\right] \\ & -\tfrac{1}{2}\mathrm{Tr}\left[\mathbf{Z}\dot{\mathbf{G}}_0^T\boldsymbol{\Sigma}^T\left\{\mathbf{1}-\mathbf{G}^T\boldsymbol{\Sigma}^T\right\}^{-1}\right].\end{aligned}}$$

- with second variational derivative:

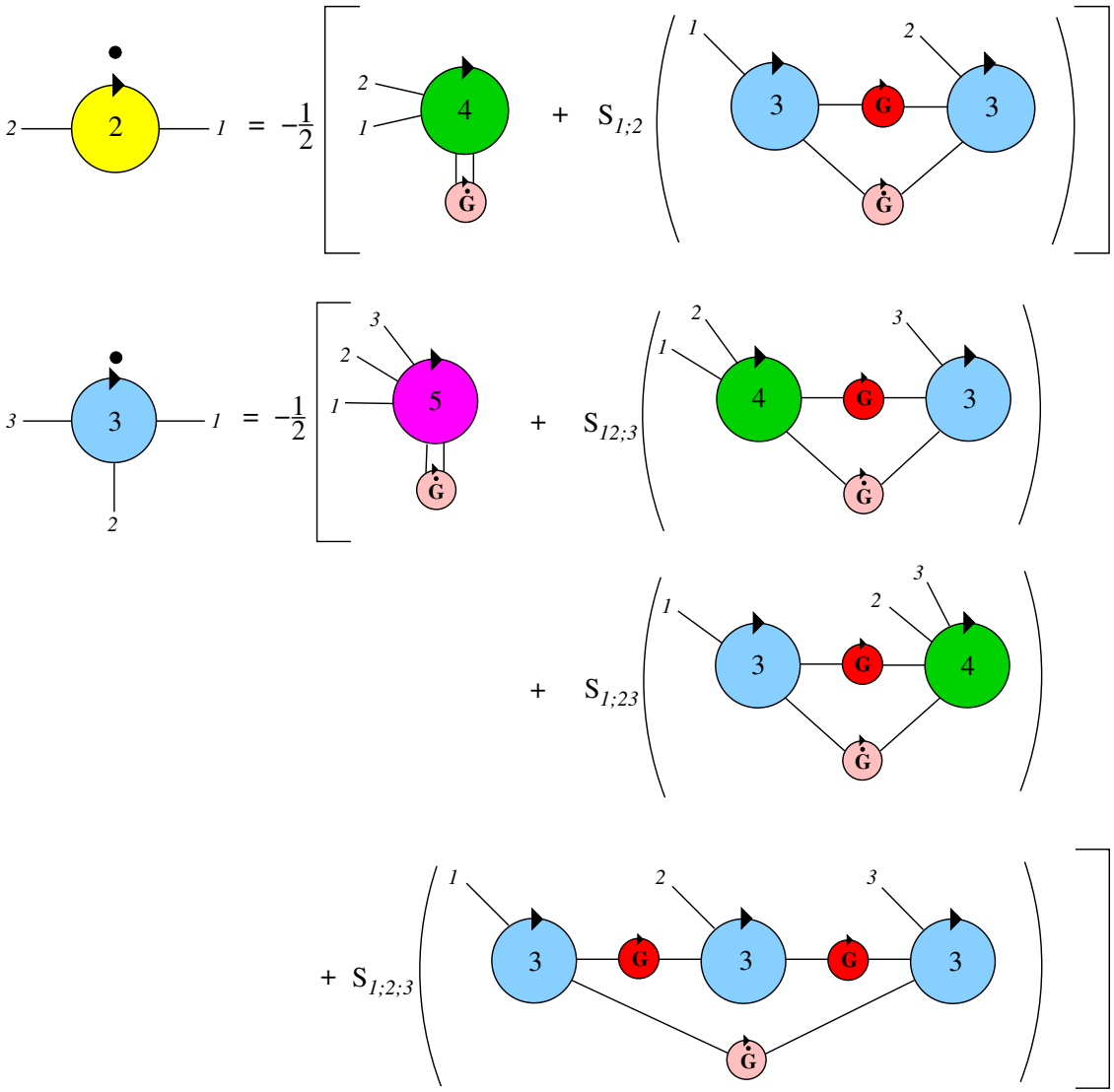
$$\mathbf{U}^T := \frac{\delta^2\Gamma}{\delta\Phi\delta\Phi} - \left.\frac{\delta^2\Gamma}{\delta\Phi\delta\Phi}\right|_{\Phi=0} = \frac{\delta^2\Gamma}{\delta\Phi\delta\Phi} - \boldsymbol{\Sigma}^T,$$

- and single scale propagator:

$$\dot{\mathbf{G}} = -\mathbf{G}\partial_\Lambda[\mathbf{G}_0^{-1}]\mathbf{G}.$$

2...flow of one-line irreducible vertices

- expand Γ in powers of Φ
- diagrammatics:



3. Exact RG for the Tomonaga-Luttinger Model

- Question:

are there non-trivial examples where infinite hierachy of coupled integro-differential equations can be solved exactly?

- YES! (F. Schütz, L. Bartosch, P.K.: PRB 2005)

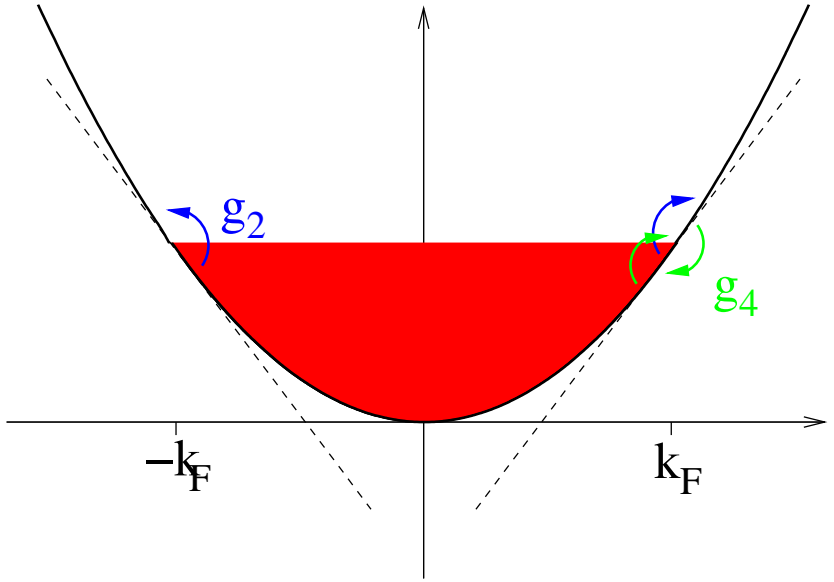
interacting electrons with dominant forward scattering and linear energy dispersion (Tomonaga-Luttinger model)

- strategy:

1. decouple electron-electron interaction in the zero-sound channel via Hubbard-Stratonovich transformation.
2. derive exact infinite hierarchy of RG flow equations for coupled Fermi-Bose field theory
3. solution is given by infinitely many by Ward identities

3...interacting Fermions in 1d

- interaction can be parameterized in terms of four marginal couplings
  - g_2: forward scattering, opposite Fermi points
  - g_4: forward scattering, same Fermi point
  - g_1: backward scattering
  - g_3: Umklapp scattering (commensurate fillings)



- RG flow in coupling space has been discussed for many years (Solyom, 1979; field theory RG).
- no RG calculations of single-particle Green function
- Tomonaga-Luttinger model: forward scattering: g_2, g_4, linear dispersion
- exact calculation of $G(x,t)$ via bosonization (Luther and Peschel, 1974)

3...Hubbard-Stratonovich-Transformation

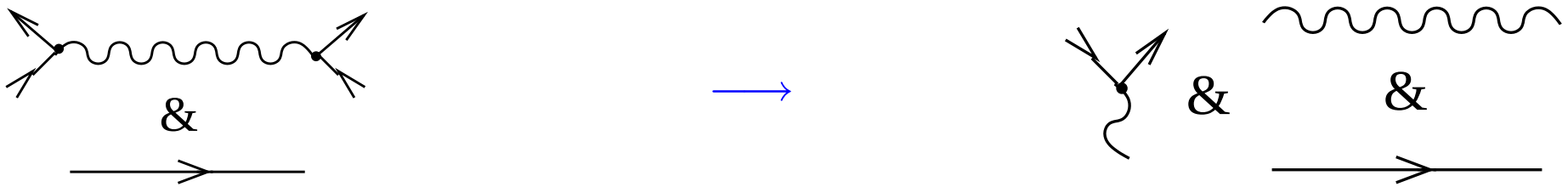
- starting point: Euclidean action with only density-density interactions:

$$S[\psi] = \sum_{\sigma} \int_{K} \bar{\psi}_{K\sigma}[-i\omega + \xi_k]\psi_{K\sigma} + \frac{1}{2}\sum_{\sigma\sigma'} \int_{\bar{K}} f_{\bar{k}}^{\sigma\sigma'} \bar{\rho}_{\bar{K}\sigma}\rho_{\bar{K}\sigma'}$$

- fermonic density operators: $\rho_{\bar{K}\sigma} = \int_K \bar{\psi}_{K\sigma}\psi_{K+\bar{K},\sigma}$

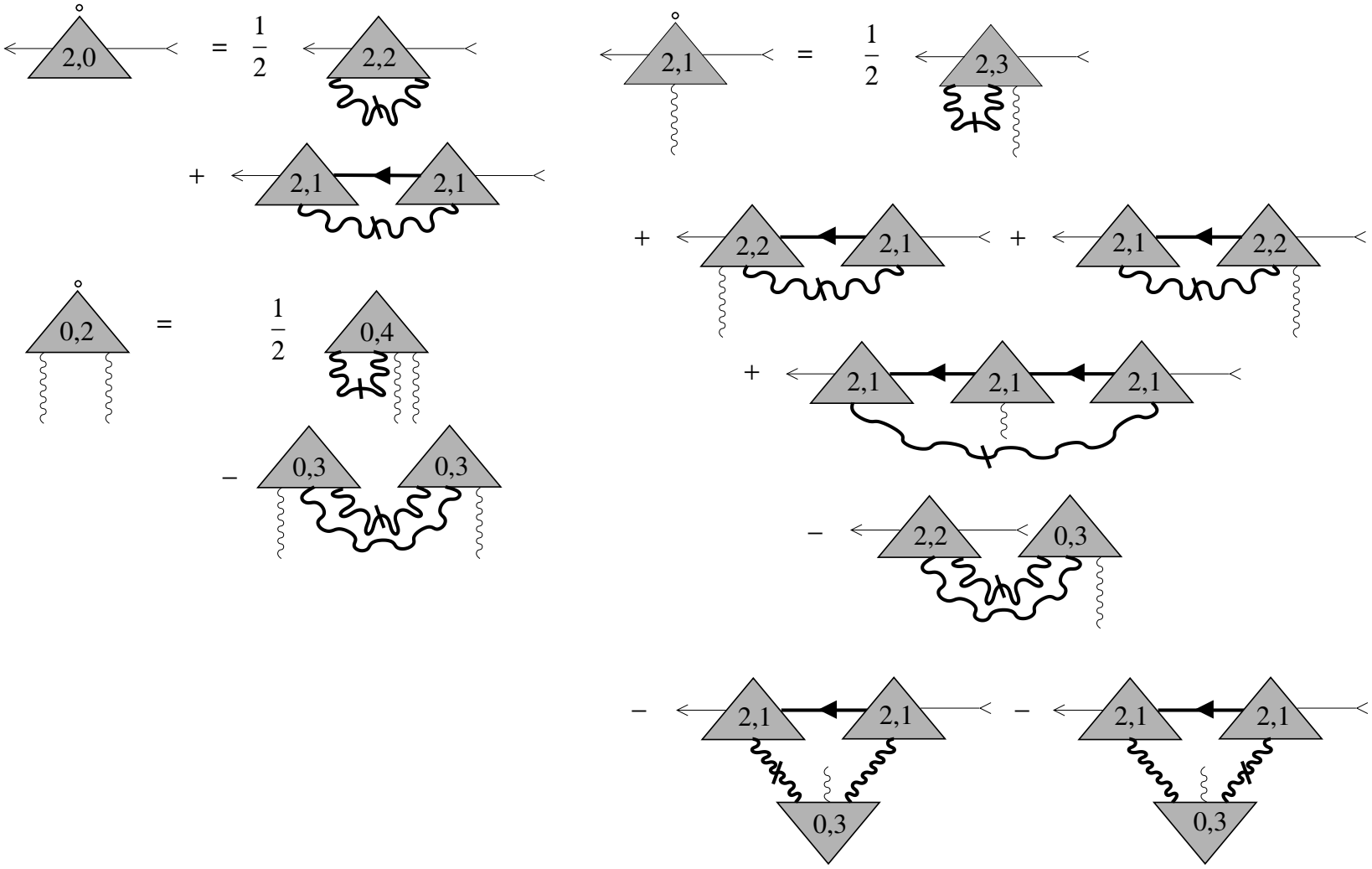
$\Downarrow$ Hubbard-Stratonovich transformation

$$S[\psi,\varphi] = \sum_{\sigma} \int_{K} \bar{\psi}_{K\sigma}[-i\omega + \xi_k]\psi_{K\sigma} + \frac{1}{2}\sum_{\sigma\sigma'} \int_{\bar{K}} [f_{\bar{k}}^{-1}]^{\sigma\sigma'} \varphi^{*}_{\bar{K}\sigma}\varphi_{\bar{K}\sigma'} + i\sum_{\sigma} \int_{\bar{K}} \bar{\rho}_{\bar{K}\sigma}\varphi_{\bar{K}\sigma}\,.$$



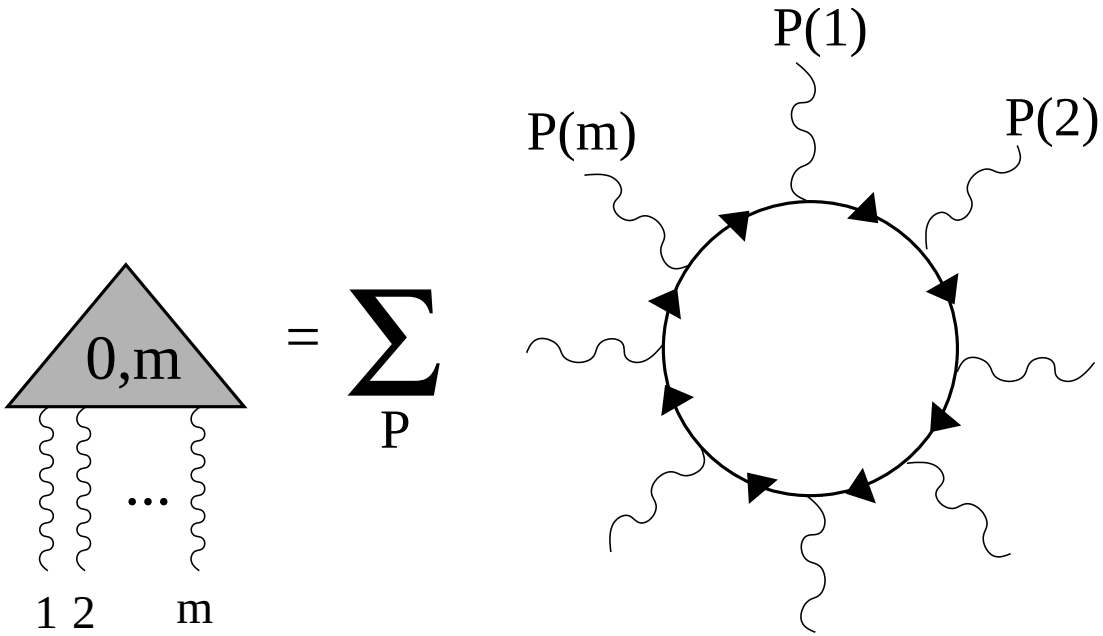
3...momentum transfer cutoff scheme

- use super-field method with $\Phi = (\psi, \bar{\psi}, \varphi)$
- cutoff only in bosonic field • exact flow equations:

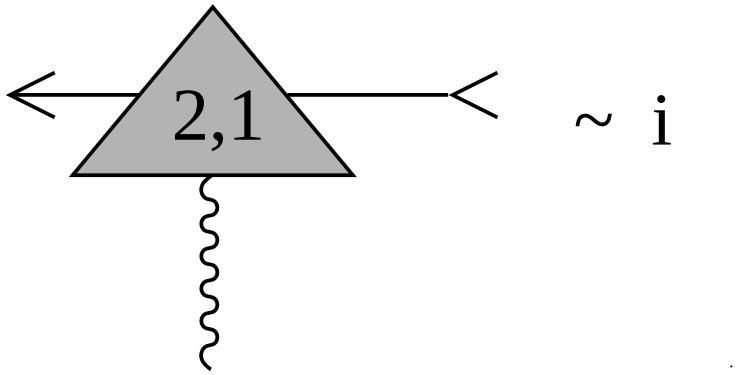


3...initial condition

- Fermi loops



- bare vertex:



- all other vertices vanish

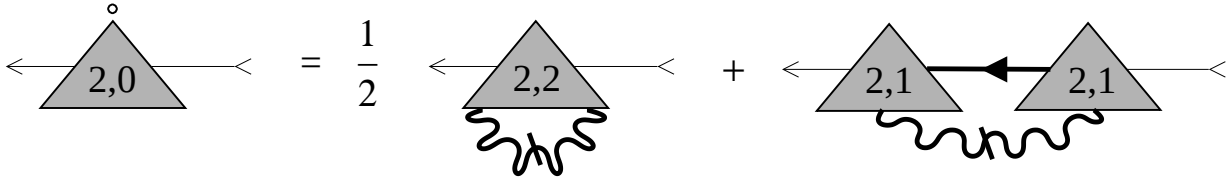
- linear dispersion: closed-loop theorem

$$\Gamma^{(0,m)}\Big|_{\Lambda=\Lambda_0} = 0\,, \qquad m > 2\,.$$

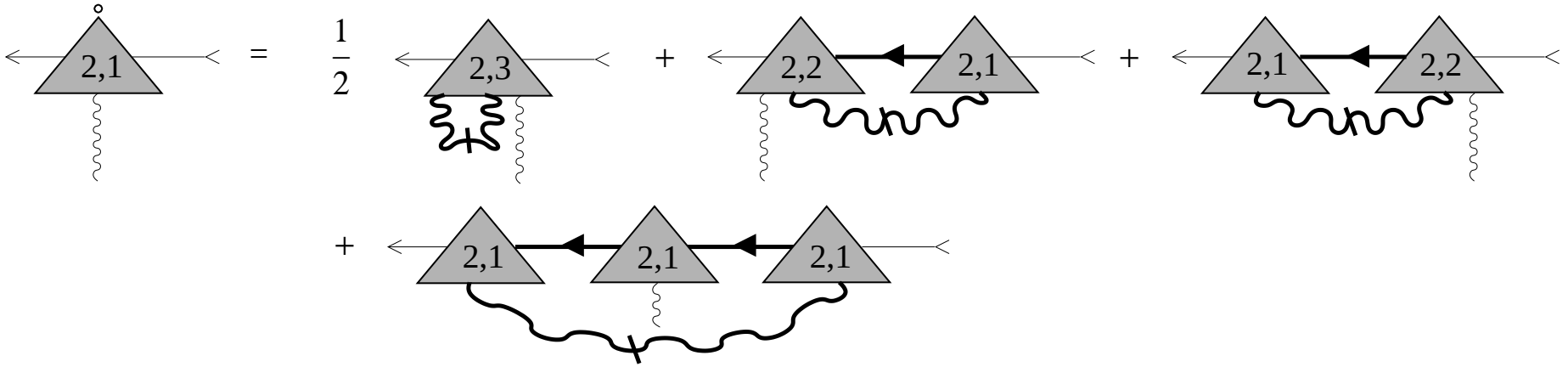
$\Rightarrow$ pure boson vertices don't flow

$$\partial_\Lambda \Gamma^{(0,m)} = 0\,.$$

3...RG flow without higher boson vertices



attach an additional Boson leg in all possible ways



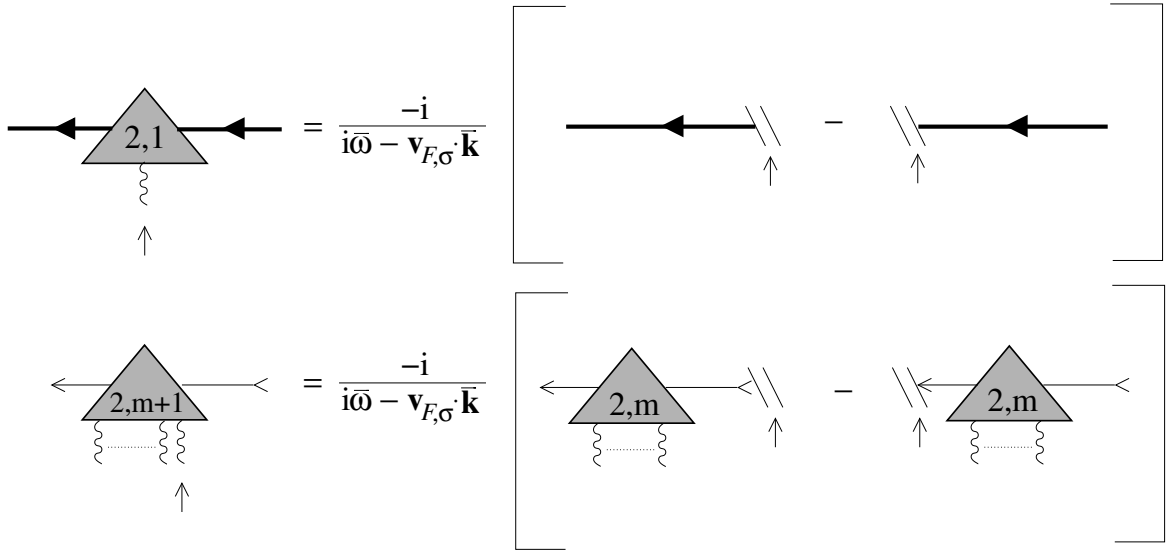
- simple structure, solve complete via Ward identities!

3...Ward identities as solution of flow equations

- Ward identity for vertex correction:

$$G(K+\bar{K})\Gamma^{(2,1)}(K+\bar{K};K;\bar{K})G(K) = \frac{-i}{i\bar{\omega}-\mathbf{v}_{F,\sigma}\cdot\mathbf{k}}\Big[G(K+\bar{K})-G(K)\Big]$$

- diagrammatically:



consistent with flow equations:

- ids trivially fulfilled initially
- insert into flow for $\Gamma^{(2,m+1)}$
- intermediate 'breaks' cancel out
- difference of flow of $\Gamma^{(2,m)}$ remains
- ids are conserved

□

$\Rightarrow$ WIs are valid at every stage of RG flow

3...exact Green-function of the TLM

- Ward ids in flow $\Rightarrow$ closed flow equation for self-energy:
- linear integro-differential equation for Green function:

$$\partial_\Lambda G_\sigma(K) = \int_{\bar{K}} \frac{\dot{F}_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \mathbf{v}_{F,\sigma}\cdot\bar{\mathbf{k}})^2}\left[G_\sigma(K) - G_\sigma(K+\bar{K})\right]$$

- solve exactly via Fourier transformation,solution as in bosonization:

$$\boxed{G_\sigma(X) = G_{\sigma,0}(X)\, e^{Q_\sigma(X)}}$$

- interaction is contained in Debye-Waller factor

$$Q_\sigma(X) = \int_{\bar{K}} \frac{\theta(|\bar{k}| < \Lambda_0) F_{\sigma\sigma}(\bar{K})}{(i\bar{\omega} - \mathbf{v}_{F,\sigma}\cdot\bar{\mathbf{k}})^2}\left[1 - \cos(\bar{\omega}\tau - \bar{\mathbf{k}}\cdot\mathbf{x})\right]$$

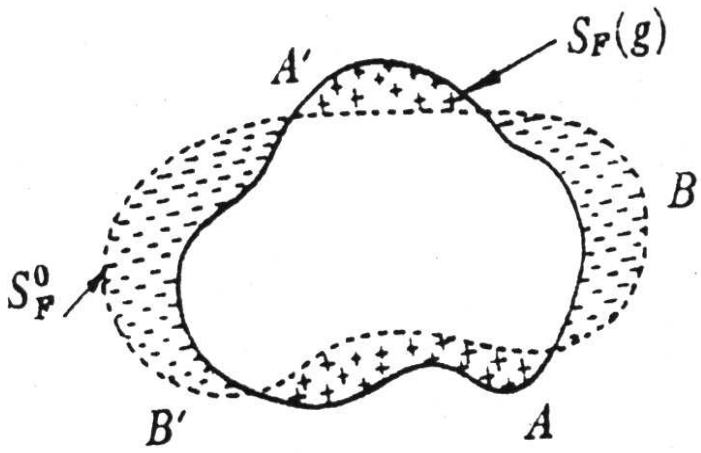
4. The Fermi surface as a RG fixed point

- Definition of the Fermi surface:

$$\varepsilon_{\mathbf{k}} + \Sigma(\mathbf{k}, i0) - \mu = 0$$

Problems:

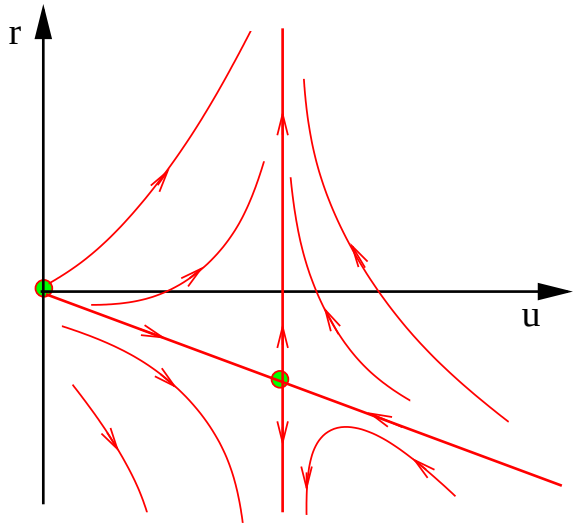
- Self-energy $\Sigma(\mathbf{k}_F, i0)$ not known a priori
- Fermi surface can change shape due to interaction



Nozières, 1964:
“In practice, we shall never try to calculate the Fermi surface, which is much too difficult. What is more important to us is to know that it exists.”

4....RG fixed point manifolds

- analogy with ϕ^4-theory: flow of r_l and u_l



- fine tune initial value r_0 to hit fixed point
- fixed point manifold zero-dimensional; gapless dispersion at fixed point
- classical system with higher-dimensional fixed point manifold: Brazovskii universality class: [S. A. Brazosvkii, Sov. Phys. JETP 41, 85 (1975)]
  - fluctuation spectrum has minimum at surface in $\mathbf{k}$-space
  - large phase for fluctuations in direction transverse to surface
- Quantum system with higher-dimensional fixed point manifold: normal Fermions in $D > 1$: Fermi surface = RG fixed point manifold
  - need fine tuning of continuum of relevant parameters $r_l(\mathbf{k}_F)$
  - What is $r_l(\mathbf{k}_F)$?

4... exact RG equation for the Fermi surface

- P. K., T. Busche, Phys. Rev. B 64, 155101 (2001)
- S. Ledowski and P. K., J. Phys.: Condens. Matter 15, 4770 (2003)
- S. Ledowski, P.K., A. Ferraz, Phys. Rev. B 71, 235106 (2005)

- continuum of relevant parameters that fix Fermi surface

$$r_l(\mathbf{k}_F) = \frac{Z_l(\mathbf{k}_F)}{v_F \Lambda}\left[\Sigma_\Lambda(\mathbf{k}_F, i0) - \Sigma(\mathbf{k}_F, i0)\right]$$

  - $\Sigma(\mathbf{k}_F, i0)$ = true self-energy at true FS
  - $\Lambda = \Lambda_0 e^{-l}$ = RG running cutoff
  - $\Sigma_\Lambda(\mathbf{k}_F, i0)$ = flowing self-energy at true FS
  - $Z_l^{\mathbf{k}_F}$ = wave function renormalization factor

- exact RG flow equation

$$\partial_l r_l(\mathbf{k}_F) \;=\; [1 - \eta_l(\mathbf{k}_F)] r_l(\mathbf{k}_F) + \dot{\Gamma}_l^{(2)}(\mathbf{k}_F) \;,\; \eta_l(\mathbf{k}_F) = -\partial_l \ln Z_l(\mathbf{k}_F)$$

$$\dot{\Gamma}_l^{(2)}(\mathbf{k}_F) = \int_{\mathbf{k}'_F} \int \frac{dq' d\varepsilon'}{(2\pi)^2} \frac{\delta(1-|q'|)\tilde{\Gamma}_l^{(4)}(\mathbf{k}_F, Q'; Q', \mathbf{k}_F)}{\left[Z_l(\mathbf{k}'_F)\left[i\varepsilon' - v'_F q'\right] - \tilde{\Gamma}^{(2)}(Q')\right]} \;,$$

  - q rescaled momenta $\mathbf{k} = \mathbf{k}_F + \hat{\mathbf{v}}_F \Lambda q$ ε: rescaled frequencies $i\omega = i v_F \Lambda \varepsilon$,
  - $\tilde{\Gamma}_l^{(2)}(Q) = \frac{Z_l(\mathbf{k}_F)}{v_F \Lambda}[\Sigma_\Lambda(\mathbf{k} = \mathbf{k}_F + \hat{\mathbf{v}}_F \Lambda q, i\omega = i v_F \Lambda \varepsilon) - \Sigma_\Lambda(\mathbf{k}_F, i0)]$, $Q = (\mathbf{k}_F, q, i\varepsilon)$.

4...reconstructing the counter-term

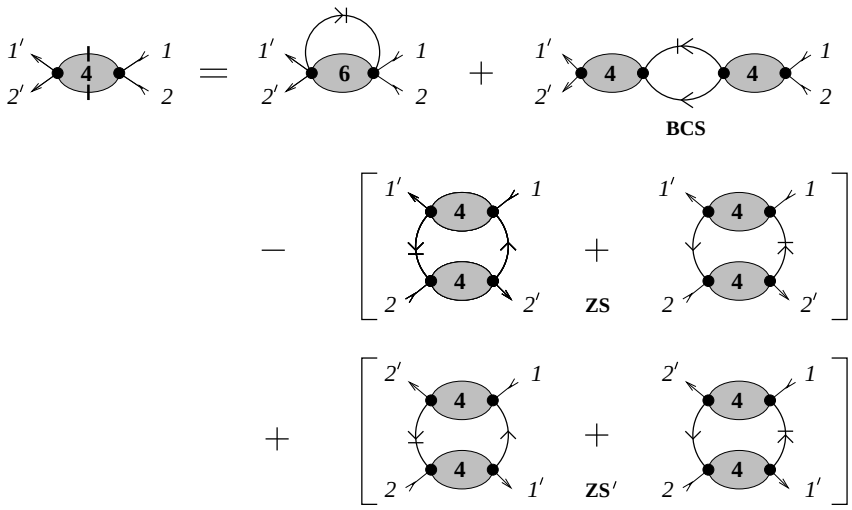
- Requirement that $r_l(\mathbf{k}_F)$ flows into fixed point yields exact integral equation for true Fermi surface:

$$r_0(\mathbf{k}_F) = -v_F \Lambda_0 \Sigma(\mathbf{k}_F, i0) = -\int_0^\infty dl e^{-l+\int_0^l dt \eta_t(\mathbf{k}_F)} \dot{\Gamma}_l^{(2)}(\mathbf{k}_F)$$

$$= \int_0^\infty dl e^{-l+\int_0^l dt \eta_t(\mathbf{k}_F)} \int_{\mathbf{k}'_F} \int \frac{dq' d\varepsilon'}{(2\pi)^2} \frac{\delta(1-|q'|)\tilde{\Gamma}_l^{(4)}(\mathbf{k}_F, Q'; Q', \mathbf{k}_F)}{\left[Z_l(\mathbf{k}'_F)[i\varepsilon' - v'_F q'] - \tilde{\Gamma}_l^{(2)}(Q')\right]}$$

- relates initial value $r_0(\mathbf{k}_F)$ to RG flow of all couplings

- flow of four-point vertex:

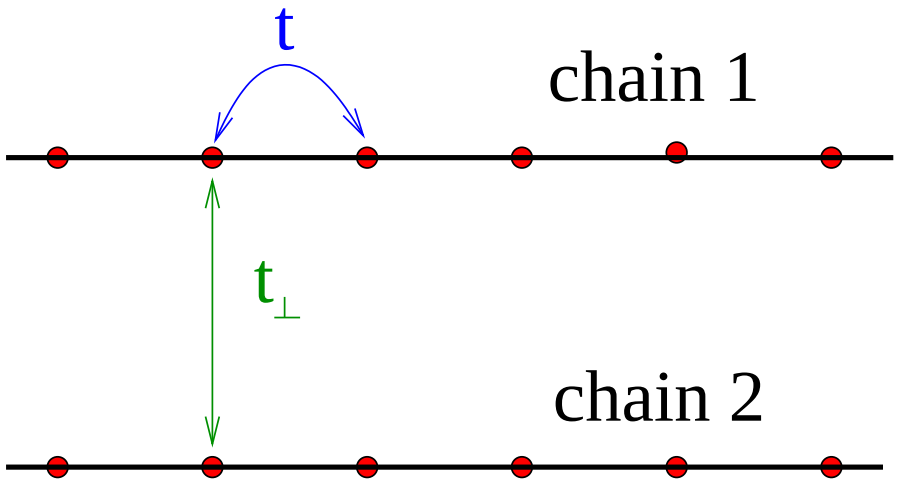


4... example: two coupled Luttinger liquids

- kinetic energy:

$$\hat{H}_0 = -t \sum_{i=1}^{N} \sum_{\sigma=1,2} (c^{\dagger}_{i,\sigma} c_{i+1,\sigma} + h.c.)$$

$$-t_{\perp} \sum_{i=1}^{N} (c^{\dagger}_{i,1} c_{i,2} + c^{\dagger}_{i,2} c_{i,1})$$

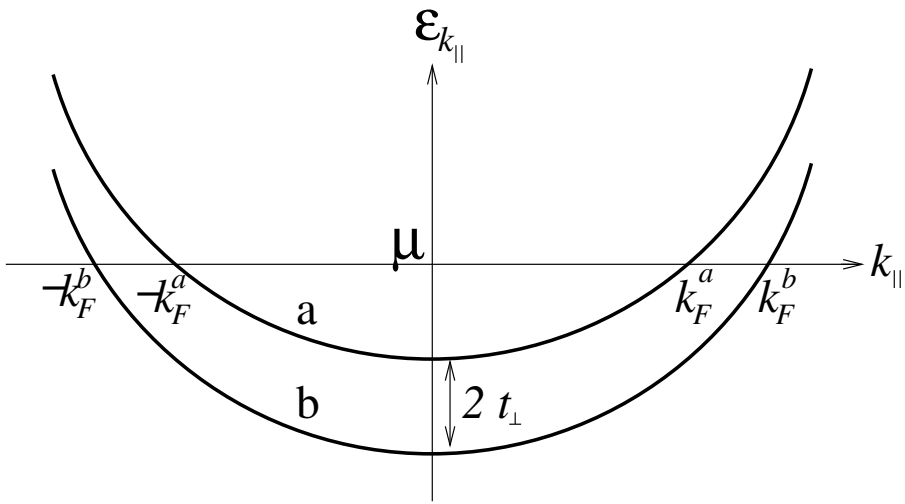


- band structure:

2 bands (bonding, antibonding)

4 Fermi points

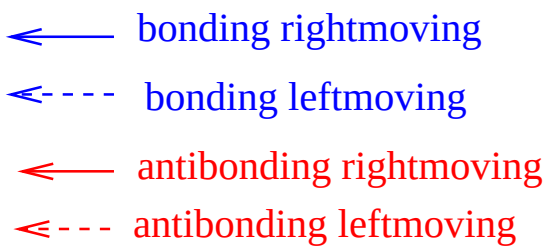
total density $n = (k_F^a + k_F^b)/\pi$



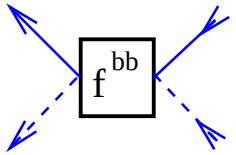
4...interactions in two LLs

$$S_{int} = \frac{1}{(2!)^2}\int_{K_1'}\int_{K_2'}\int_{K_2}\int_{K_1} \times \sum_{\sigma_1'\sigma_2'\sigma_2\sigma_1} \delta_{K_1'+K_2',K_2+K_1} \times \Gamma_{\Lambda_0}(K_1'\sigma_1', K_2'\sigma_2'; K_2\sigma_2, K_1\sigma_1) \times \overline{\psi}_{K_1'\sigma_1'}\overline{\psi}_{K_2'\sigma_2'}\psi_{K_2\sigma_2}\psi_{K_1\sigma_1}$$

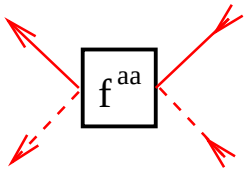
- ignore chiral vertices
- interband backward scattering for $k_F^b \neq k_F^a$ not momentum conserving on FS
- momentum labels $\mathbf{k} = \alpha k_F^\sigma + \Lambda q$, chirality index $\alpha = \pm 1$.



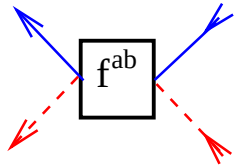
bonding forward:



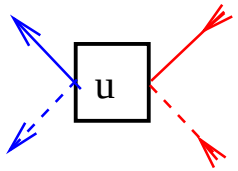
antibonding forward :



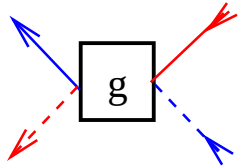
interband forward:



interband Umklapp :
(pair tunneling)



interband backward :



4. ...vertex functions in two coupled LLs

$$\tilde{\Gamma}_l^{(4)}(\sigma_1'\alpha_1'Q_1', \sigma_2'\alpha_2'Q_2'; \sigma_2\alpha_2Q_2, \sigma_1\alpha_1Q_1) = (2\pi v_F)^{-1}$$
$$\times \Big(\prod_{i=1,2} Z_l^{\sigma_i'} Z_l^{\sigma_i}\Big)^{1/2} \Gamma_\Lambda^{(4)}(\sigma_1'K_1', \sigma_2'K_2'; \sigma_2K_2, \sigma_1K_1) \ , \ \alpha_i = \pm 1 \text{(chirality index)}$$

need functional RG flow of five vertex functions:

$$\tilde{f}_l^{bb}(Q_1', Q_2'; Q_2, Q_1) = \tilde{\Gamma}_l^{(4)}(bQ_1', bQ_2'; bQ_2, bQ_1),$$
$$\tilde{f}_l^{aa}(Q_1', Q_2'; Q_2, Q_1) = \tilde{\Gamma}_l^{(4)}(aQ_1', aQ_2'; aQ_2, aQ_1),$$
$$\tilde{f}_l^{ab}(Q_1', Q_2'; Q_2, Q_1) = \tilde{\Gamma}_l^{(4)}(aQ_1', bQ_2'; bQ_2, aQ_1),$$
$$\tilde{u}_l(Q_1', Q_2'; Q_2, Q_1) = \tilde{\Gamma}_l^{(4)}(bQ_1', bQ_2'; aQ_2, aQ_1),$$
$$\tilde{g}_l(Q_1', Q_2'; Q_2, Q_1) = \tilde{\Gamma}_l^{(4)}(bQ_1', aQ_2'; bQ_2, aQ_1),$$

(chirality labels are $(\alpha_1', \alpha_2'; \alpha_2, \alpha_1) = (+, -; -, +)$)

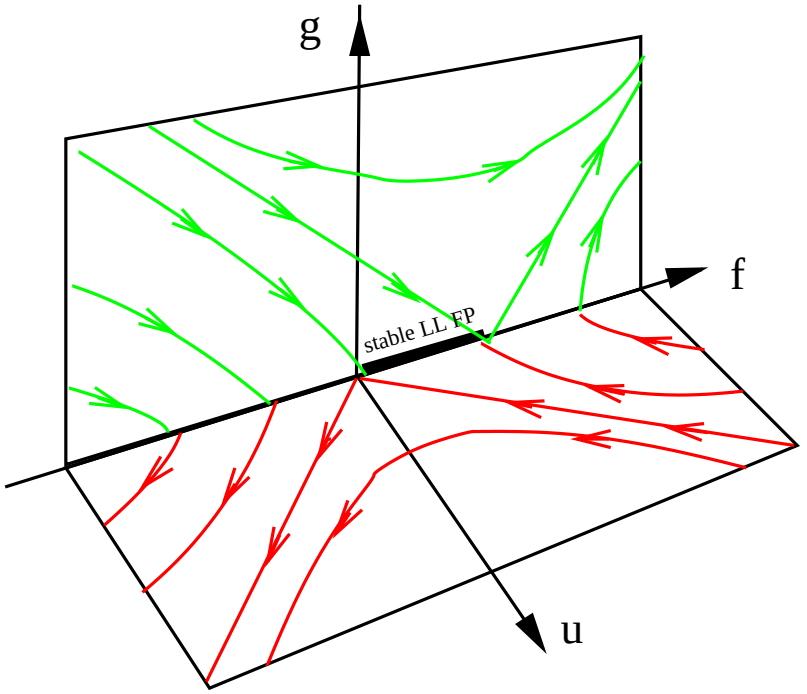
4..flow of coupling constants

Flow of momentum/frequency independent part of vertices:

$$
\begin{aligned}
f_l^{bb} &= \tilde{f}_l^{bb}(Q_1', Q_2'; Q_2, Q_1)\big|_{Q_i=0} \\
f_l^{aa} &= \tilde{f}_l^{aa}(Q_1', Q_2'; Q_2, Q_1)\big|_{Q_i=0} \\
f_l^{ab} &= \tilde{f}_l^{ab}(Q_1', Q_2'; Q_2, Q_1)\big|_{Q_i=0} \\
u_l &= \tilde{u}_l(Q_1', Q_2'; Q_2, Q_1)\big|_{Q_i=0} \\
g_l &= \tilde{g}_l(Q_1', Q_2'; Q_2, Q_1)\big|_{Q_i=0}
\end{aligned}
$$

one-loop RG flow equations:

$$
\begin{aligned}
\partial_l u_l &= -u_l f_l \\
\partial_l g_l &= g_l f_l \\
\partial_l f_l &= -A u_l^2 + B_l g_l^2 \\
f_l &= f_l^{aa}/v_0^a + f_l^{bb}/v_0^b - 2 f_l^{ab}/\bar{v}_0
\end{aligned}
$$



gapless Luttinger liquid phase stable in finite parameter regime

4....one-loop flow of vertex functions

$$
\begin{aligned}
D_l^{bb}\,\tilde{f}_l^{bb}(Q_1',Q_2';Q_2,Q_1) &= (f_l^{bb})^2[\dot{\chi}_l^{bb}(Q_1-Q_2')-\dot{\chi}_l^{bb}(Q_1+Q_2)] \\
&\quad -u_l^2\dot{\chi}_l^{aa}(Q_1+Q_2)+g_l^2\dot{\chi}_l^{aa}(Q_1-Q_2'+2\tilde{\Delta}_l)\;, \\
D_l^{aa}\,\tilde{f}_l^{aa}(Q_1',Q_2';Q_2,Q_1) &= \text{similar expression} \\
D_l^{ab}\,\tilde{f}_l^{ab}(Q_1',Q_2';Q_2,Q_1) &= \text{similar expression} \\
D_l^{ab}\tilde{u}_l(Q_1',Q_2';Q_2,Q_1) &= f_l^{ab}u_l[\dot{\chi}_l^{ab}(Q_1-Q_2')+\dot{\chi}_l^{ba}(Q_1-Q_2')] \\
&\quad -f_l^{bb}u_l\dot{\chi}_l^{bb}(Q_1+Q_2)-f_l^{aa}u_l\dot{\chi}_l^{aa}(Q_1+Q_2)\;, \\
D_l^{ab}\tilde{g}_l(Q_1',Q_2';Q_2,Q_1) &= -f_l^{ab}g_l[\dot{\chi}_l^{ab}(Q_1+Q_2)+\dot{\chi}_l^{ba}(Q_1+Q_2)] \\
&\quad +f_l^{bb}g_l\dot{\chi}_l^{bb}(Q_1-Q_2')+f_l^{aa}g_l\dot{\chi}_l^{aa}(Q_1-Q_2')\;.
\end{aligned}
$$

$$
D_l^{\alpha\beta} = \partial_l+\eta_l^{\alpha}+\eta_l^{\beta}+\sum_{i=1}^{2}(Q_i\cdot\partial_{Q_i}+Q_i'\cdot\partial_{Q_i'})\;,\quad \tilde{\Delta}_l=(k_F^b-k_F^a)e^l/\Lambda_0
$$

$$
\dot{\chi}_l^{\alpha\beta}(Q) = \frac{\Theta(e^l-1-|q|)}{2\bar{v}_0+v_0^{\alpha}|q|-i\varepsilon\mathrm{sgn}q}+\frac{\Theta(e^l-1-|q|)}{2\bar{v}_0+v_0^{\beta}|q|+i\varepsilon\mathrm{sgn}q}\;,
$$

4...flow of self-energy in Luttinger liquid phase

$$\partial_l \tilde{\Gamma}_l^{(2)}(\sigma\alpha Q) = (1-\eta_l^{\sigma} - Q\cdot\partial_Q)\tilde{\Gamma}_l^{(2)}(\sigma\alpha Q) + \dot{\Gamma}_l^{(2)}(\sigma\alpha Q)\ ,$$

$$\begin{aligned}\dot{\Gamma}_l^{(2)}(a,+,Q) &= f_l^{aa} + f_l^{ab} - \partial_l[f_l^{aa} + f_l^{ab}] - 2(f_l^{aa})^2 I_{a,l}^{aa}(Q) - 2(f_l^{ab})^2 I_{b,l}^{ab}(Q) \\ &\quad -u_l^2[I_{a,l}^{bb}(Q) + I_{b,l}^{ba}(Q)] - g_l^2[I_{a,l}^{bb}(q-2\tilde{\Delta}_l, i\varepsilon) + I_{b,l}^{ba}(q-2\tilde{\Delta}_l, i\varepsilon)]\ ,\end{aligned}$$

$$\begin{aligned}I_{\gamma,l}^{\alpha\beta}(Q) &= \frac{1}{v_0^{\alpha}+v_0^{\beta}} \sum_{n=\pm 1} n\Theta(e^l - 1 - |q+n|) \\ &\quad \times \left\{ \Theta(n\,\mathrm{sgn}(q+n)) \ln\left[\frac{e^l(v_0^{\alpha}+v_0^{\beta}) + v_0^{\gamma} - v_0^{\beta}|q+n| - in\varepsilon}{v_0^{\alpha}+v_0^{\beta}+v_0^{\gamma}+v_0^{\alpha}|q+n| - in\varepsilon}\right]\right. \\ &\quad \left. +\Theta(-n\,\mathrm{sgn}(q+n)) \ln\left[\frac{e^l(v_0^{\alpha}+v_0^{\beta}) + v_0^{\gamma} - v_0^{\alpha}|q+n| - in\varepsilon}{v_0^{\alpha}+v_0^{\beta}+v_0^{\gamma}+v_0^{\beta}|q+n| - in\varepsilon}\right]\right\}\end{aligned}$$

4...renormalized Fermi surface

- counter-terms from initial value r_0^σ:

$$\Sigma(\sigma, \alpha k_F^\sigma, i0) = -v_F \Lambda_0 r_0^\sigma$$
$$= \int_0^\infty dl e^{-l+\int_0^l dt \eta_t^\sigma} \dot{\Gamma}_l^{(2)}(\sigma, \alpha, 0)$$

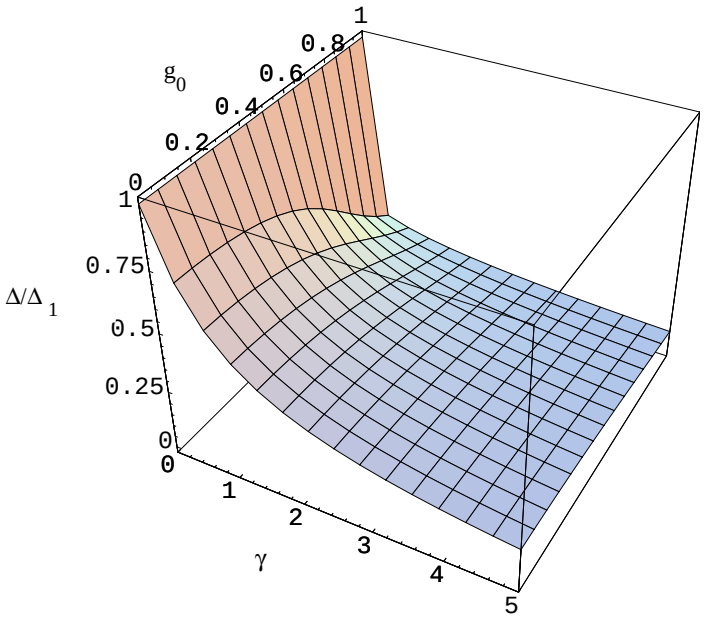
Self-consistency condition

$$\Delta = \Delta_1 [1 + g_0^2 \ln(\Lambda_0/\Delta)^2]^{-1}$$

Δ_1 = Hartree-Fock result,
$\gamma = g_0^2 \ln(\Lambda_0/\Delta_1)^2$.

- to leading logarithmic order:

$$r_0^a = -f_0^{aa} - f_0^{ab} - g_0^2 \tilde{\Delta} \ln|\tilde{\Delta}|$$
$$r_0^b = -f_0^{bb} - f_0^{ba} + g_0^2 \tilde{\Delta} \ln|\tilde{\Delta}|$$



- Fermi surface distance $\Delta = k_F^b - k_F^a$ at constant density $n = (k_F^a + k_F^b)/\pi$:

$$\Delta = \Delta_0 + 2\Lambda_0 (v_0^b + v_0^a)^{-1} (r_0^b - r_0^a)$$

5. Vacuum expectation values

- Florian Schütz, P.K., to appear in cond-mat in the first week of December

- What happens if field has finite vacuum expectation value?

$$\left.\frac{\delta \mathcal{G}_c[J]}{\delta J_\alpha}\right|_{J=0} = \langle \Phi_\alpha \rangle|_{J=0} \equiv \Phi^0_\alpha \neq 0 \,.$$

- Define generating functional with shifted field $\Delta\Phi = \Phi - \Phi^0$

$$\Gamma[\Phi] = \mathcal{L}\,[\Phi] + \frac{1}{2}(\Delta\Phi, \mathbf{G}_0^{-1}\Delta\Phi) \,.$$

- Define vertices via expansion

$$\Gamma[\Phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \dots \int_{\alpha_n} \Gamma^{(n)}_{\alpha_1,\dots,\alpha_n} \Delta\Phi_{\alpha_1} \cdot \dots \cdot \Delta\Phi_{\alpha_n} \,.$$

5. ...vacuum expectation values

- Extra terms in flow equation (tadpole diagrams)

$$\begin{aligned}\partial_\Lambda \Gamma &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \dots \int_{\alpha_n} \partial_\Lambda \Gamma^{(n)}_{\alpha_1,\dots,\alpha_n} \Delta\Phi_{\alpha_1} \cdot \dots \cdot \Delta\Phi_{\alpha_n} \\ &- \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha} \int_{\alpha_1} \dots \int_{\alpha_n} \Gamma^{(n+1)}_{\alpha\alpha_1,\dots,\alpha_n} (\partial_\Lambda \Phi^0_\alpha) \Delta\Phi_{\alpha_1} \cdot \dots \cdot \Delta\Phi_{\alpha_n} \ .\end{aligned}$$

- modified flow of free energy:

$$\partial_\Lambda \Gamma^{(0)} = -\frac{1}{2} \mathrm{Tr} \left[\mathbf{Z} \dot{\mathbf{G}}_0^T \Sigma^T \left\{ \mathbf{1} - \mathbf{G}_0^T \Sigma^T \right\}^{-1} \right] - \frac{1}{2} (\Phi^0, [\partial_\Lambda \mathbf{G}_0^{-1}] \Phi^0) - \int_\alpha (\partial_\Lambda \Phi_\alpha) \Gamma^{(1)}_\alpha \ .$$

- Flow of vertex with one external leg:

$$\partial_\Lambda \Gamma^{(1)}_{\alpha_1} = -\frac{1}{2} \mathrm{Tr} \left[\mathbf{Z} \dot{\mathbf{G}}^T \Gamma^{(3)T}_{\alpha_1} \right] - \int_\alpha [\mathbf{G}^{-1}]_{\alpha_1 \alpha} \partial_\Lambda \Phi^0_\alpha - \int_\alpha [\partial_\Lambda \mathbf{G}_0^{-1}]_{\alpha_1 \alpha} \Phi^0_\alpha \ ,$$

5. ...vacuum expectation values

- Extra terms in flow equation (tadpole diagrams)

$$\begin{aligned}\partial_\Lambda \Gamma &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha_1} \dots \int_{\alpha_n} \partial_\Lambda \Gamma^{(n)}_{\alpha_1,\dots,\alpha_n} \Delta\Phi_{\alpha_1} \cdot \dots \cdot \Delta\Phi_{\alpha_n} \\ &- \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\alpha} \int_{\alpha_1} \dots \int_{\alpha_n} \Gamma^{(n+1)}_{\alpha\alpha_1,\dots,\alpha_n} (\partial_\Lambda \Phi^0_\alpha) \Delta\Phi_{\alpha_1} \cdot \dots \cdot \Delta\Phi_{\alpha_n} \ .\end{aligned}$$

- modified flow of free energy:

$$\partial_\Lambda \Gamma^{(0)} = -\frac{1}{2} \mathrm{Tr} \left[\mathbf{Z} \dot{\mathbf{G}}_0^T \Sigma^T \left\{ \mathbf{1} - \mathbf{G}_0^T \Sigma^T \right\}^{-1} \right] - \frac{1}{2} (\Phi^0, [\partial_\Lambda \mathbf{G}_0^{-1}] \Phi^0) - \int_\alpha (\partial_\Lambda \Phi_\alpha) \Gamma^{(1)}_\alpha \ .$$

- Flow of vertex with one external leg:

$$\partial_\Lambda \Gamma^{(1)}_{\alpha_1} = -\frac{1}{2} \mathrm{Tr} \left[\mathbf{Z} \dot{\mathbf{G}}^T \Gamma^{(3)T}_{\alpha_1} \right] - \int_\alpha [\mathbf{G}^{-1}]_{\alpha_1 \alpha} \partial_\Lambda \Phi^0_\alpha - \int_\alpha [\partial_\Lambda \mathbf{G}_0^{-1}]_{\alpha_1 \alpha} \Phi^0_\alpha \ ,$$

5. ...flow of vacuum expectation value

- Simplifications
  - choose $\mathbf{G}_0^{-1}\Phi^0 = 0$.
  - require $\Gamma_\Lambda^{(1)} = 0$: no tadpole diagrams.
- flow equation for vacuum expectations value:

$$\int_\alpha [\Sigma]_{\alpha_1\alpha}\partial_\Lambda \Phi^0_\alpha = \frac{1}{2}\mathrm{Tr}\left[\Gamma^{(3)}_{\alpha_1}\dot{\mathbf{G}}\right] \ ,$$

- Application:
  - classical φ^4-theory (see gradient expansion, effective potential).
  - interacting Bose gas in condensed phase (truncation of the exact flow equation that preserves symmetries)
  - flow of compressibility in interacting Fermi system.

Summary, Outlook

Summary:

- General functional RG method using super-fields
- electrons in 1d with forward scattering: exact solution of fRG
  - momentum transfer cutoff in the interaction
  - closed loop theorem as initial condition
  - Ward identities to close flow equations
- Fermi surface problem
  - exact integral equation for true Fermi surface
  - explicit calculation for two coupled chains

Outlook:

- truncation schemes when Ward ids are not valid
- decoupling in other channels
- broken symmetry in Fermi systems: order parameter=bosonic field
- quantum impurity models