Theory of rigid-body motion

Hendrik van Hees

December 17, 2021

Contents

1 Introduction

The motion of rigid bodies is described in almost all standard textbooks on classical mechanics (e.g., [Som94, Gre03, LL60]). Usually it is considered challenging for undergraduate students in the first theoretical-physics course lecture on classical mechanics. In this paper we aim at a simplification of the treatment at the cost to use more advanced mathematical tools as the properties of the rotation group, SO(3), as a Lie group and Hamilton's least-action principle, which we consider justified by the fact that these advanced mathematical tools have to be introduced anyway.

The paper is organized as follows: In Sect. 2 we introduce into the description of the rigid-body dynamics with an intuitive argument, leading to Euler's theorem that the rigid body is described by six independent degrees of freedom, descring the translative motion of one body-fixed point relative to the origin of an inertial reference frame (henceforth called the "space-fixed frame") as well as an arbitrary rotation of an arbitrary body-fixed Cartesian system of basis vectors relative to the Cartesian basis system of the space-fixed frame (describing the orientation of the rigid body relative to the space-fixed frame). The analysis of the possible motions of the system, described by a position vector $\vec{R}$ pointing from the space-fixed origin to an arbitrary body-fixed point and a rotation matrix $\hat{D} \in \mathrm{SO}(3)$, expressing the body-fixed Cartesian basis vectors as superposition of the space-fixed Cartesian basis vectors, which leads in a natural way to the notion of “infinitesimal rotations” and the description by axial vectors, particularly the momentaneous angular velocity $\vec{\omega}$ of the rigid body’s rotation around the origin of the body-fixed reference frame.

In Sect. 3 we address the dynamics of the rigid body by deriving the expression for the kinetic energy as well as the potential energy taking into account the homogeneous gravitational field of the Earth to derive the Lagrange function, leading to a natural definition of the center of mass (as one special important body-fixed reference point) and the tensor of inertia.

In Sect. 4 we derive the Euler equations of motion first without introducing specific parametrizations of the rotation matrix transforming from body-fixed to space-fixed vector (and tensor) components but apply directly Hamilton’s least-action principle by using variations of the rotation matrix.

In Sect. 5 we introduce the Euler angles and use the experience with the rotation group to derive in an efficient way the angular-velocity vector components with respect to both the body-fixed and the space-fixed reference frame and finally apply in Sect. 5 the Hamilton principle to the analysis of the usual topics considered in textbooks, as the stability analysis of the rotations of a asymmetric free top, the motion of the symmetric free top, as well as the heavy symmetric top, which is first discussed qualitatively and then giving a complete analytical solution in the small-oscillation approximation for the “fast-spinning top”.

In the Appendix we give an elegant continuum-mechanical proof of Euler’s theorem, used in Sect. 2 to introduce the rigid-body dynamics in an intuitive way.

2 Kinematics of a rigid body

In Newtonian mechanics a rigid body is introduced as an extended solid piece of matter, for which the relative position of any two of its points stays fixed. Intuitively it is clear that the motion of this body can be described with respect to an arbitrary “space-fixed” inertial reference frame with origin O by (a) fixing an arbitrary point O' in the body and (b) the orientation of an arbitrary body-fixed Cartesian basis system $\vec{e}_k'$ (with $k \in \{1,2,3\}$) relative to the space-fixed Cartesian basis system $\vec{e}_j$ (with $j \in \{1,2,3\}$).

This relative orientation of the $\vec{e}_k'$ to the $\vec{e}_j$ is given by a matrix $\hat{D} = (D_{jk})$ such that

$$\vec{e}_k' = \vec{e}_j D_{jk}, \tag{1}$$

where we use the Einstein summation convention, according to which in any expression, where two equal indices occur, one has to sum over this index pair from 1 to 3.

Since both basis systems are Cartesian, i.e.,

$$\vec{e}_k' \cdot \vec{e}_l' = \vec{e}_k \cdot \vec{e}_l = \delta_{kl} = \begin{cases} 1 & \text{for} \quad k = l, \\ 0 & \text{for} \quad k \neq l, \end{cases} \tag{2}$$

the matrix elements of $\hat{D}$ are given by

$$D_{jk} = \vec{e}_j \cdot \vec{e}_k' \tag{3}$$

and from this we get

$$D_{jk}D_{jl} = (\vec{e}_j \cdot \vec{e}_k')(\vec{e}_j \cdot \vec{e}_l') = \vec{e}_k' \cdot \vec{e}_l' = \delta_{kl}, \quad D_{jk}D_{lk} = (\vec{e}_j \cdot \vec{e}_k')(\vec{e}_l \cdot \vec{e}_k') = \vec{e}_j \cdot \vec{e}_l = \delta_{jl}. \tag{4}$$

In matrix notation this means

$$\hat{D}^{\mathrm{T}}\hat{D} = \hat{D}\hat{D}^{\mathrm{T}} = \hat{1} \Leftrightarrow \hat{D}^{-1} = \hat{D}^{\mathrm{T}}, \tag{5}$$

which defines a orthogonal matrix. It is also easy to prove that the set of orthogonal matrices form a (non-Abelian) group, the Orthogonal Group O(3) of the Euclidean vector space $\mathbb{R}^3$.

Now we want to restrict the matrix $\hat{D}$ further by assuming that it describes a continuous motion $\hat{D}(t)$, i.e., the change of the body-fixed basis relative to the space-fixed basis as a function of time. To that end we further assume that both Cartesian bases are right-handed bases, i.e.,

$$(\vec{e}_j \times \vec{e}_k) \cdot \vec{e}_l = (\vec{e}_j' \times \vec{e}_k') \cdot \vec{e}_l' = \epsilon_{jkl}, \tag{6}$$

where the Levi-Civita symbol ϵ_{jkl} is defined to be totally antisymmetric under interchange of any pair of indices and $\epsilon_{123} = 1$. From this we obtain

$$\epsilon_{jkl} = (\vec{e}_j' \times \vec{e}_k') \cdot \vec{e}_l' = D_{aj}D_{bk}D_{cl}(\vec{e}_a \times \vec{e}_b) \cdot \vec{e}_c = D_{aj}D_{bk}D_{cl}\epsilon_{abc} = \epsilon_{jkl} \det \hat{D} \Rightarrow \det \hat{D} = 1. \tag{7}$$

Thus fixing the orientation of both basis systems to be right-handed leads to $\det \hat{D} = 1$, and all orthogonal matrices with this property form a subgroup of the O(3), the Special Orthogonal Group, the SO(3).

Now we can charcterize the motion of the rigid body as follows: We specify the location of the body-fixed origin by the position vector $\vec{R}(t) = \overrightarrow{OO'}$ and the relative orientation of the body by the SO(3) matrix $\hat{D}(t)$ which determines the body-fixed basis in terms of the space-fixed basis by (1).

It is now intuitive that an SO(3) matrix corresponds to three degrees of freedom since an arbitrary rotation can be characterized by a unit vector $\vec{n}$ (which can be parametrized by two angles $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$ as in spherical coordinates) and a rotation angle $\phi \in [0, 2\pi)$, where the rotation is given by the right-hand rule: pointing with the thumb of the right hand in the direction of $\vec{n}$ the fingers indicate the sense of rotation.

It is also intuitive that any rotation around a fixed rotation axis $\vec{n}$, which we shall write as $\hat{D}(\vec{n}, \phi)$ can composed of very many rotations around a small angle $\delta\phi = \phi/N$, where N is some very large natural number. This leads to the idea to write

$$\hat{D}(\vec{n}, \phi + \delta\phi) = (\hat{1} + \delta\phi\hat{\Omega})\hat{D}(\vec{n}, \phi). \tag{8}$$

Since this should be again an SO(3) matrix, we must have

$$\begin{aligned}
\hat{1} &= \hat{D}(\vec{n}, \phi + \delta\phi)\hat{D}^{\mathrm{T}}(\vec{n}, \phi + \delta\phi) \\
&= (\hat{1} + \delta\phi\hat{\Omega})\hat{D}(\vec{n}, \phi)\hat{D}^{\mathrm{T}}(\vec{n}, \phi)(\hat{1} + \delta\phi\hat{\Omega})^{\mathrm{T}} \\
&= (\hat{1} + \delta\phi\hat{\Omega})(\hat{1} + \delta\phi\hat{\Omega}^{\mathrm{T}}) \\
&= \hat{1} + \delta\phi(\hat{\Omega}) + \hat{\Omega}^{\mathrm{T}}) + \mathcal{O}(\delta\phi^2).
\end{aligned} \tag{9}$$

This implies that at linear order in $\delta\phi$ we must have

$$\hat{\Omega} + \hat{\Omega}^{\mathrm{T}} = 0 \Leftrightarrow \Omega_{jk} = -\Omega_{kj}, \tag{10}$$

i.e., the "infinitesimal generator" of a rotation in direction of $\vec{n}$ must be an anti-symmetric matrix, which can be written via a vector $\vec{\omega}$ through

$$\Omega_{jk} = -\epsilon_{jkl}\omega_l. \tag{11}$$

A rotation with its axis along $\vec{n}$ does not change this vector, i.e.,

$$0 = \Omega_{jk} n_k = -\epsilon_{jkl} n_k \omega_l \Rightarrow \omega_l = n_l, \tag{12}$$

i.e., the infinitesimal rotation transforming body-fixed vector components to space-fixed components of an arbitrary vector $\vec{x}$, according to (1),

$$x_j = D_{jk}(\vec{n}, \delta\phi) x'_k = x'_j - \delta\phi \epsilon_{jkl} n_l x_k = x'_j + \delta\phi \epsilon_{jlk} n_l x'_k \Leftrightarrow \underline{x} = \underline{x}' + \delta\phi \underline{n} \times \underline{x}', \tag{13}$$

where $\underline{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ and $\underline{x}' = (x'_1, x'_2, x'_3)^{\mathrm{T}}$ denote the space- and body-fixed components of the vector $\vec{x}$ written as an $\mathbb{R}^3$-column-vector.

Finally we note that ϵ_{jkl} transforms like invariant tensor components under SO(3) transformations since

$$\epsilon'_{jkl} = D_{ja} D_{kb} D_{lc} \epsilon_{abc} = \epsilon_{jkl} \det \hat{D} = \epsilon_{jkl}. \tag{14}$$

This implies that Ω_{jk} transform as 2^{nd}-rank tensor components since the n_l transform as vector components under SO(3) transformations.

The above analysis of "infinitesimal rotations" is a simple example for a Lie group, which can be characterized as a matrix group which smoothly depends on a parameter. We shall briefly show that we can indeed calculate the finite rotation around a fixed axis $\vec{n}$ from the above derived description of infinitesimal rotations by integrating the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\phi} \hat{D}(\vec{n}, \phi) = \hat{\Omega} \hat{D}(\vec{n}, \phi), \quad \vec{n} = \text{const}, \tag{15}$$

which follows from (8) taking the limit $\delta\phi \to 0$. A formal solution is easily found, using the matrix-exponential function. Assuming $\hat{D}(\vec{n}, \phi = 0) = \hat{1}$ we get by formal integration

$$\hat{D}(\vec{n}, \phi) = \exp(\hat{\Omega}\phi) = \sum_{k=0}^{\infty} \frac{1}{k!} (\hat{\Omega}\phi)^k. \tag{16}$$

To evaluate this exponential explicitly it is sufficient to consider its action on an arbitrary column vector $\underline{x}'$. Using (11) we get

$$\phi \hat{\Omega} \underline{x}' = \phi \underline{n} \times \underline{x}', \quad (\phi \hat{\Omega})^2 \underline{x}' = \phi^2 \underline{n} \times (\underline{n} \times \underline{x}') = \phi^2 [\underline{n}(\underline{n} \cdot \underline{x}) - \underline{x}], \quad (\hat{\Omega}\phi)^3 \underline{x}' = -\phi^3 \underline{n} \times \underline{x}', \ldots \tag{17}$$

By induction we can easily prove that

$$(\hat{\Omega}\phi)^{2j+1} = \phi^{2j+1} (-1)^j \underline{n} \times \underline{x}, \quad (\hat{\Omega}\phi)^{2j} = \phi^{2j} (-1)^{j+1}. \tag{18}$$

Plugging this into the sum in (16) leads to

$$\hat{D}(\vec{n}, \phi) \underline{x}' = \underline{n}(\underline{n} \cdot \vec{x}) + \sin\phi \, \underline{n} \times \underline{x} - \cos\phi \, \underline{n} \times (\underline{n} \times \underline{x}') = \underline{n}(\underline{n} \cdot \vec{x}) + \sin\phi \, \underline{n} \times \underline{x}' + \cos\phi \, [\underline{x}' - \underline{n}(\underline{n} \cdot \underline{x}')]. \tag{19}$$

Now we consider the rigid body as a set of point masses m_k held together by massless rigid rods. This is somewhat artificial, and of course in practice we shall use a continuum-mechanical description as detailed in the Appendix. For the purpose of the derivation of the kinematics and dynamics of the rigid body this discrete picture is somewhat more intuitive given that one can stay within the concept of point-particle mechanics.

Now we can describe any point P_k of the rigid body by the position vector wrt. to the space-fixed reference frame $\overrightarrow{OP_k} = \vec{x}_k = \vec{x}_k(t)$ and decompose it as

$$\vec{x}_i(t) = \vec{R}(t) + \vec{r}_i(t), \tag{20}$$

where $\vec{R}(t) = \overrightarrow{OO'}$ is the position vector of the arbitrary body-fixed point O' with respect to the origin of the space-bound reference frame and $\vec{r}_k = \overrightarrow{O'P_k}$ the position vector of P_k in the body-fixed reference frame.

The rigidity condition is then easily described by the fact that the components of all $\vec{r}_k$ with respect to the body-fixed basis are time-independent, i.e.,

$$\underline{r}_k(t) = \hat{D}(t)\underline{r}'_k \quad \text{with} \quad \underline{r}'_k = \text{const.} \tag{21}$$

3 The dynamics

To describe the dynamics of the rigid body we use Hamilton's principle of least action, according to which the equations of motion are given as the trajectories $q(t)$ with fixed points $q(t_1)$ and $q(t_2)$ (where $q = (q_1, \ldots, q_f)$ are arbitrary generalized coordinates to describe the independent degrees of freedom of a mechanical system), for which the action functional

$$S[q] = \int_{t_1}^{t_2} \mathrm{d}t L(q, \dot{q}) \tag{22}$$

gets minimal (or at least stationary). Variation around this minimizing trajectory leads to

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \left[\delta q_k \frac{\partial L}{\partial q_k} + \delta \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} \right] = \int_{t_1}^{t_2} \mathrm{d}t \, \delta q_k \left[\frac{\partial L}{\partial q_k} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_k} \right] \stackrel{!}{=} 0, \tag{23}$$

where in the last step if have integrated by parts, using $\delta q(t_1) = \delta q(t_2) = 0$. Since this equation should hold for all functions $\delta q_k(t)$, the bracket under the integral must vanish, leading to the Euler-Lagrange equations

$$p_k := \frac{\partial L}{\partial \dot{q}_k}, \quad \dot{p}_k = \frac{\partial L}{\partial q_k}. \tag{24}$$

Here we introduced the generalized momenta p_k. It is easy to see that for a system of particles interacting via some conservative forces the equations of motion are given by the Lagrange function

$$L = T - V = \sum_k \frac{m_k}{2} \dot{\vec{x}}_k^2 - V(\vec{x}_1, \ldots, \vec{x}_N). \tag{25}$$

Here the $\vec{x}_k$ have to be read as functions of the independent generalized coordinates $(q_1, \ldots, q_f)$.

For our rigid body we choose as the generalized coordinates the three space-frame components $\underline{R}$ of the vector $\vec{R}$ and the SO(3) matrix $\hat{D}$ transforming from body-frame to space-frame components of arbitrary vectors $\vec{V}$ via $\underline{V} = \hat{D}\underline{V}'$. As we have seen above, we can think of this matrix as parametrized by three independent rotational degrees of freedom (like the two components of the momentaneous rotation-axis unit-direction vector $\vec{n}$ and the momentaneous rotation angle or, more convenient for the description of the rigid body, the Euler angles to be introduced in Sect. 5). As we shall see now, we can abstain from introducing a specific parametrization of this rotation matrix for quite a while, thanks to the flexibility of Hamilton's principle.

To that end we start with deriving the kinetic energy of the rigid body, we use (20) and the fact that the space-frame basis $\vec{e}_j$ is time-independent to evaluate

$$\dot{\vec{x}}_k^2 = \dot{\underline{x}}_k^2 = (\dot{\underline{R}} + \dot{\underline{r}}_k)^2. \tag{26}$$

Now the body-frame components $\underline{r}'_k$ of the $\vec{r}_k$ are time-independent and thus

$$\dot{\underline{r}}_k = \frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{D}\underline{r}'_k\right) = \dot{\hat{D}}\underline{r}'_k = \dot{\hat{D}}\hat{D}^{\mathrm{T}}\underline{r}_k. \tag{27}$$

Now since $\hat{D} \in \mathrm{SO}(3)$

$$\hat{D}\hat{D}^{\mathrm{T}} = \hat{1} \Rightarrow \hat{\Omega} = \dot{\hat{D}}\hat{D}^{\mathrm{T}} = -\hat{D}\dot{\hat{D}}^{\mathrm{T}} = -\left(\dot{\hat{D}}\hat{D}^{\mathrm{T}}\right)^{\mathrm{T}} = -\hat{\Omega}^{\mathrm{T}}, \tag{28}$$

and from that

$$\dot{\underline{r}}_k = \hat{\Omega}\underline{r}_k = \underline{\omega} \times \underline{r}_k, \tag{29}$$

where as argued in the similar case of (11)

$$\Omega_{ab} = -\epsilon_{abc}\omega_c. \tag{30}$$

Using

$$\epsilon_{abc}\epsilon_{ade} = \delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd} \tag{31}$$

we obtain from (30) the inverse relation

$$\omega_c = -\frac{1}{2}\Omega_{ab}\epsilon_{abc}. \tag{32}$$

We note that $\vec{\omega}$ is the momentaneous angular velocity of the rotation of the body-frame basis relative to the space-frame basis, describing an infinitesimal transformation $\mathrm{d}\hat{\Omega} = \mathrm{d}t\dot{\phi}\vec{n}$, where $\vec{n} = \vec{\omega}/|\vec{\omega}|$ and $\dot{\phi} = |\vec{\omega}|$. Now plugging (29) into (26) yields

$$\dot{\vec{x}}_k^2 = \dot{\vec{R}}^2 + 2\dot{\vec{R}} \cdot (\vec{\omega} \times \vec{r}_k) + (\vec{\omega} \times \vec{r}_k)^2. \tag{33}$$

Finally using this in (25) results in

$$T = \sum_k \frac{m_k}{2}\dot{\vec{x}}_k^2 = \frac{M}{2}\dot{\vec{R}}^2 + M\dot{\vec{R}} \cdot (\vec{\omega} \times \vec{s}) + \frac{1}{2}\sum_k m_k(\vec{\omega} \times \vec{r}_k)^2 \tag{34}$$

with the total mass and the center-of-mass position vector $\vec{s}$ in the body-fixed frame defined by

$$M = \sum_k m_k, \quad \vec{s} = \frac{1}{M} \sum_k m_k \vec{r}_k. \tag{35}$$

The three terms in (34) can be interpreted as the kinetic energy of the translational motion of the body as a whole, the kinetic energy through the rotation of the body's center of mass around the body-fixed reference point O', and finally the kinetic energy of the spin of the body around the body-fixed reference point. We note that the choice of the body-centered reference point as well as the orientation of both the space-fixed and the body-fixed basis can, dependent on the concrete problem to be solved, simplify (34) considerably. To be flexible, we keep the most general choice in (34). While the translational and the "spin-orbit part" of the kinetic theory are already in a useful form, the spin part is most easily expressed in terms of the body-fixed vector components since the $\underline{r}'_k$ are time-independent. To that end we first write

$$(\vec{\omega} \times \vec{r}_k)^2 = (\vec{\omega} \times \vec{r}_k) \cdot (\vec{\omega} \times \vec{r}_k) = \vec{\omega} \cdot [\vec{r}_k \times (\vec{\omega} \times \vec{r}_k)] = \vec{\omega}[\vec{\omega}\vec{r}_k^{\,2} - \vec{r}_k(\vec{\omega} \cdot \vec{r}_k) = \vec{\omega}^{\,2}\vec{r}_k^{\,2} - (\vec{r}_k \cdot \vec{\omega})^2. \tag{36}$$

Writing the body-fixed components of $\vec{r}_k$ as $r_j^{\prime(k)}$, the spin part of the kinetic energy can finally be brought into the form

$$T_{\text{spin}} = \sum_k \frac{m_k}{2} (\vec{\omega} \times \vec{r}_k)^2 = \sum_k \frac{m_k}{2} \omega'_a \omega'_b (\underline{r}_k^{\prime 2} \delta_{ab} - r_a^{\prime(k)} r_b^{\prime k}) = \frac{1}{2} \Theta'_{ab} \omega'_a \omega'_b. \tag{37}$$

This defines the body-fixed components of the symmetric tensor of inertia, which are time-independent, as

$$\Theta'_{ab} = \Theta'_{ba} = \sum_k m_k (\underline{r}_k^{\prime 2} \delta_{ab} - r_a^{\prime(k)} r_b^{\prime k}). \tag{38}$$

As is well known from linear algebra, any symmetric real matrix can be diagonalized with an SO(3) transformation. Since $\hat{\Theta}' = (\Theta'_{ab})$ is time-independent the corresponding rotation matrix is also time-independent and can be simply considered as a transformation from the originally chosen right-handed body-frame Cartesian basis to another such body-frame basis. In the following we assume that the body-fixed basis is chosen in direction of the principle axes of the tensor of inertia, such that $\hat{\Theta}' = \operatorname{diag}(A, B, C)$ with $A, B, C > 0$.

In the following we shall consider only the constant gravitational force of the Earth acting on the rigid body. The potential energy is

$$V = -\sum_k m_k \vec{g} \cdot \vec{x}_k = -\sum_k m_k \vec{g}(\vec{R} + \vec{r}_k) = -M\vec{g} \cdot (\vec{R} + \vec{s}). \tag{39}$$

The Lagrangian thus finally reads, using (34), (37), and (39)

$$L = T - V = \frac{M}{2} \dot{\vec{R}}^2 + M\dot{\vec{R}} \cdot (\vec{\omega} \times \vec{s}) + \frac{1}{2} \underline{\omega}'^{\mathrm{T}} \hat{\Theta}' \underline{\omega}' + M\vec{g} \cdot (\vec{R} + \vec{s}). \tag{40}$$

In the following we shall derive the equations of motion for the usual cases of the spinning top, where one point of the body is fixed, which we choose as the origin of both the space- and body-fixed reference frames. Then $\vec{R} = 0 = \text{const}$, and the Lagrangian simplifies to

$$L = \frac{1}{2} \underline{\omega}'^{\mathrm{T}} \hat{\Theta}' \underline{\omega}' + M\vec{g} \cdot \vec{s}. \tag{41}$$

4 Euler's Equations of the spinning top

To derive Euler's equations of motion for the spinning top as equations for the body-fixed angular velocity components $\underline{\omega}'$ we need these components in terms of the rotation matrix $\hat{D}$ between the space- and body-fixed bases. To that end we use (28) and not that $\hat{\Omega}$ transforms as tensor components. Since for vectors we have $\underline{V} = \hat{D}\underline{V}'$ or $\underline{V}' = \hat{D}^{\mathrm{T}}\underline{V}$ thus

$$\hat{\Omega}' = \hat{D}^{\mathrm{T}}\hat{\Omega}\hat{D} = \hat{D}^{\mathrm{T}}\dot{\hat{D}}\hat{D}^{\mathrm{T}}\hat{D} = \hat{D}^{\mathrm{T}}\dot{\hat{D}}. \tag{42}$$

and from that, analogous to (30),

$$\Omega'_{ab} = -\epsilon_{abc}\omega'_c. \tag{43}$$

To derive the equations of motion we use $\hat{D}$ as the generalized coordinates and write the variation as

$$\delta\hat{D} = \hat{D}\delta\hat{K}, \tag{44}$$

where $\delta\hat{K} = -\delta\hat{K}^{\mathrm{T}}$, so that the variation is compatible to order $\mathscr{O}(\delta\hat{K})$ with the contraint that $\hat{D}$ and $\hat{D} + \delta\hat{D}$ are both SO(3) matrices. Thus we can write

$$\delta K_{ab} = -\delta K_{ba} = \epsilon_{abc}\delta k_c \tag{45}$$

with independent δk_c. The variation of the kinetic part of the Lagrangian (41) is

$$\delta T = \hat{\Theta}'\delta\underline{\omega}', \tag{46}$$

where we made use of the symmetry $\hat{\Theta}'^{\mathrm{T}} = \hat{\Theta}'$ of the tensor of inertia. To evaluate $\delta\omega'$ we use

$$\delta\hat{\Omega}' = \delta\hat{D}^{\mathrm{T}}\dot{\hat{D}} + \hat{D}^{\mathrm{T}}\delta\dot{\hat{D}} = \delta\hat{K}^{\mathrm{T}}\hat{D}^{\mathrm{T}}\dot{\hat{D}} + \hat{D}^{\mathrm{T}}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{D}\delta\hat{K}) = -\delta\hat{K}\hat{\Omega}' + \hat{\Omega}'\delta\hat{K} + \delta\dot{\hat{K}}, \tag{47}$$

where we have used the antisymmetry of $\delta\hat{K}$ and (42). Written in the Ricci calculus this translates to

$$\delta\Omega'_{kl} = \Omega'_{ka}\delta K_{al} - \delta K_{ka}\Omega'_{al} + \delta\dot{K}_{kl} \tag{48}$$

and then, with (43)

$$\delta\omega'_n = -\frac{1}{2}\epsilon_{nkl}\delta\Omega'_{kl} = -\frac{1}{2}\epsilon_{nkl}(\delta\dot{K}_{kl} + 2\delta K_{al}\Omega_{ka}). \tag{49}$$

With (45) after some algebra we finally obtain

$$\delta\omega'_n = -\delta\dot{k}_n + \epsilon_{nkl}\omega'_l\delta k'_k. \tag{50}$$

Plugging this into (46) and integrating over t we find after integrating the term containing $\delta\dot{k}_n$ by parts

$$\delta S_{\mathrm{kin}} = \int_{t_1}^{t_2}\mathrm{d}t\Theta'_{mn}\omega'_m\delta\omega'_n = \int_{t_1}^{t_2}\mathrm{d}t\,\delta\underline{k}\cdot(\hat{\Theta}'\underline{\dot{\omega}}' + \underline{\omega}'\times\hat{\Theta}'\underline{\omega}'). \tag{51}$$

For the potential-energy part of (41) we find, again with (45) and the fact that the space-frame components $\underline{g} =$ const as well as the body-fixed components $\underline{s}' =$ const don't vary with the variation of $\hat{D}$,

$$\delta V = -M\delta(\underline{g}\cdot\underline{s}) = -m\underline{s}'\cdot\delta\hat{D}^{\mathrm{T}}\underline{g} = +M\underline{s}'\delta\hat{K}\underline{g}' = M(\underline{s}'\times\underline{g}')\delta\underline{k}. \tag{52}$$

Thus the variation of the action reads

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t\, \delta \underline{k} \cdot \left(\hat{\Theta}' \underline{\dot{\omega}}' + \underline{\omega}' \times \hat{\Theta}' \underline{\omega}' - M \underline{s}' \times \underline{g}' \right). \tag{53}$$

According to the Hamilton principle of least action, this must vanish for arbitrary $\delta \underline{k}$, leading to the Euler Equations of motion for the spinning top, written in terms of body-fixed coordinates

$$\hat{\Theta}' \underline{\dot{\omega}}' + \underline{\omega}' \times \hat{\Theta}' \underline{\omega}' = M \underline{s}' \times \underline{g}'. \tag{54}$$

To interpret this equation we note that the space-fixed components of the total angular momentum of the body (wrt. the fixed point) are

$$\underline{J} = \sum_k m_k \underline{r}_k \times \underline{\dot{r}}_k = \sum_k m_k \underline{r}_k \times (\underline{\omega} \times \underline{r}_k). \tag{55}$$

For the body-fixed components we thus have, using (29)

$$\underline{J}' = \sum_k m_k \underline{r}_k' \times (\underline{\omega}' \times \underline{r}_k') = \hat{\Theta}' \underline{\omega}'. \tag{56}$$

The latter formula follows after some algebra from the definition (38) of the body-fixed components of the inertia tensor.

Now we evaluate the time-derivative of the angular-momentum vector, using (1), (42), and (43)

$$\begin{aligned}
\dot{\vec{J}} &= \frac{\mathrm{d}}{\mathrm{d}t} (J_a' \vec{e}_a') \\
&= \dot{J}_a' \vec{e}_a' + J_a' \dot{\vec{e}}_a' = \dot{J}_a' \vec{e}_a' + J_a' \dot{D}_{ba} \vec{e}_b \\
&= \dot{J}_a' \vec{e}_a' + J_a' \dot{D}_{ba} D_{bc} \vec{e}_c' \\
&= \dot{J}_a' \vec{e}_a' + J_a' (\hat{D}^{\mathrm{T}} \dot{\hat{D}})_{ca} \vec{e}_c' \\
&= \dot{J}_a' \vec{e}_a' - J_a' \epsilon_{cab} \omega_b' \vec{e}_c' \\
&= [\dot{J}_a' + (\underline{\omega}' \times \underline{J}')_a] \vec{e}_a'.
\end{aligned} \tag{57}$$

From this we find that (54) can be written as the basis-independent equation of motion

$$\dot{\vec{J}} = M \vec{s} \times \vec{g} = \vec{\tau}, \tag{58}$$

where $\vec{\tau}$ is the torque wrt. the fixed point of the top due to the gravitational force $M\vec{g}$ applied at the center of mass.

5 Euler angles

To define the Euler angles we remind the rotations around the axes of a right-handed Cartesian basis, given by the matrices $\hat{D}^{(j)}$:

$$(\vec{e}_1', \vec{e}_2', \vec{e}_3') = (\vec{e}_1, \vec{e}_2, \vec{e}_3) \hat{D}^{(j)} \tag{59}$$

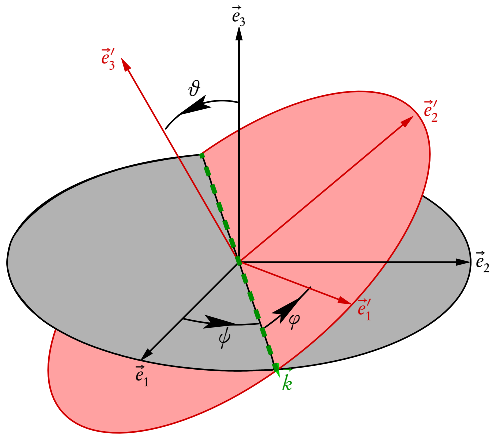


Figure 1: The definition of the Euler angles.

with

$$\hat{D}^{(1)}(\varphi)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{pmatrix},$$
$$\hat{D}^{(2)}(\varphi)=\begin{pmatrix}\cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{pmatrix}, \tag{60}$$
$$\hat{D}^{(3)}(\varphi)=\begin{pmatrix}\cos\varphi & -\sin\varphi & 0\\ \sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Now the Euler angles are introduced by performing three successive rotations such as to bring the right-handed space-fixed basis in coincidence with the body-fixed basis. First one rotates the body-fixed basis $\vec{e}_j$ around the axis $\vec{e}_3$ by an angle ψ to make the new basis vector $\vec{e}_1''$ point into the direction of the line of nodes, defined as the intersection of the plane spanned by the space-fixed basis vectors $\vec{e}_1$ and $\vec{e}_2$ (see Fig. 1), then around the line of nodes $\vec{e}_1''$ by an angle ϑ such that the $\vec{e}_3''=\vec{e}_3$ axis gets rotated into $\vec{e}_3'$ and finally around $\vec{e}_3'$ by an angle φ such that the line of nodes $\vec{e}_1''$ is rotated to $\vec{e}_1'$:

$$(\vec{e}_1',\vec{e}_2',\vec{e}_3')=(\vec{e}_1,\vec{e}_2,\vec{e}_3)\hat{D}^{(3)}(\psi)\hat{D}^{(1)}(\vartheta)\hat{D}^{(3)}(\varphi)\Rightarrow \hat{D}=\hat{D}^{(3)}(\psi)\hat{D}^{(1)}(\vartheta)\hat{D}^{(3)}(\varphi). \tag{61}$$

From this derivation of the Euler angles it is immediately clear that the three angles are only uniquely defined if $\vartheta\notin\{0,\pi\}$.

Now we need to calculate the angular velocity $\vec{\omega}$. This is easily done for the components wrt. the body-fixed frame. To that end we only have to remember that the infinitesimal rotation following from a infinitesimal change of ψ is given by

$$\underline{\omega}_\psi=\dot{\psi}\begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{62}$$

The rotation with angle ϑ is around the line of nodes, whose components are given by

$$\underline{e}_1'' = \hat{D}^{(3)}(\varphi)\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix} \Rightarrow \underline{\omega}_\vartheta = \dot{\vartheta}\begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\end{pmatrix}. \tag{63}$$

Finally the rotation wrt. ψ is around the axis

$$\underline{e}_3' = \hat{D}^{(3)}(\psi)\hat{D}^{(1)}(\vartheta)\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\vartheta\sin\psi\\ -\sin\vartheta\cos\psi\\ \cos\vartheta\end{pmatrix} \Rightarrow \underline{\omega}_\varphi = \dot{\varphi}\begin{pmatrix}\sin\vartheta\sin\psi\\ -\sin\vartheta\cos\psi\\ \cos\vartheta\end{pmatrix}. \tag{64}$$

Thus we finally get

$$\underline{\omega} = \begin{pmatrix}\dot{\vartheta}\cos\psi + \dot{\varphi}\sin\vartheta\sin\psi\\ \dot{\vartheta}\sin\psi - \dot{\varphi}\sin\vartheta\cos\psi\\ \dot{\psi} + \dot{\varphi}\cos\vartheta\end{pmatrix}. \tag{65}$$

For the body-fixed components we can invert (61),

$$(\vec{e}_1, \vec{e}_2, \vec{e}_3) = (\vec{e}_1', \vec{e}_2', \vec{e}_3')\hat{D}^{(3)\mathrm{T}}(\varphi)\hat{D}^{(1)\mathrm{T}}(\vartheta)\hat{D}^{(3)\mathrm{T}}(\psi), \tag{66}$$

and use the same method as for deriving (65) to get

$$\underline{\omega}' = \hat{D}^{\mathrm{T}}\underline{\omega} = \begin{pmatrix}\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi\\ -\dot{\vartheta}\sin\varphi + \dot{\psi}\sin\vartheta\cos\varphi\\ \dot{\varphi} + \dot{\psi}\cos\vartheta\end{pmatrix}. \tag{67}$$

Now the Lagrangian (41), using the principal axes of the inertia tensor as body-fixed bases, such that $\hat{\Theta}' = \mathrm{diag}(A, B, C)$, written in terms of the Euler angles as generalized coordinates, reads

$$\begin{aligned}L =& \frac{1}{2}\Big[A(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi)^2 + B(-\dot{\vartheta}\sin\varphi + \dot{\psi}\sin\vartheta\cos\varphi)^2 + C(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\Big]\\ &+ m\vec{g}\cdot\vec{s}.\end{aligned} \tag{68}$$

This shows that this completely general case is complicated, because even for the force-free case, where the body is fixed in its center of mass, implying $\vec{s} = 0$, only one of the Euler angles, ψ, is cyclic, which is due to rotation invariance around the space-fixed axis $\vec{e}_3$. So even in the force-free case together with energy there is only one more conserved quantity ("first integral"), and the system is not integrable.
This changes for the symmetric top, for which $B = A$. Then the Lagrangian simplifies to

$$L = \frac{1}{2}\Big[A(\dot{\vartheta}^2 + \dot{\psi}^2\sin\vartheta) + C(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\Big] + m\vec{g}\cdot\vec{s}. \tag{69}$$

Now, at least for the force-free case, besides ψ also φ is a cyclic variable, which is due to the symmetry of the body under rotations around the body-fixed symmetry axis $\vec{e}_3'$, the figure axis. That implies that the corresponding canonical momenta,

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = C(\dot{\varphi} + \dot{\psi}\cos\vartheta), \tag{70}$$

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = A\dot{\psi}\sin\vartheta + C\cos\vartheta(\dot{\varphi} + \dot{\psi}\cos\vartheta) = A\dot{\psi}\sin\vartheta + p_\varphi\cos\vartheta. \tag{71}$$

are conserved for the force-free case (and also in the case of the heavy top, since $\vec{s} = s\vec{e}_3'$ due to symmetry, and by choosing the space-fixed basis such that $\vec{g} = -g\vec{e}_3$). Since L also does not depend explicitly on time, the total energy function,

$$E = T + V = \frac{1}{2}\left[A(\dot{\vartheta}^2 + \dot{\psi}^2 \sin\vartheta) + C(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\right] - m\vec{g}\cdot\vec{s} = \text{const.} \tag{72}$$

The energy takes the simple form $T + V$ since the Lagrangian is a quadratic form in the generalized velocities $\dot{\varphi}$, $\dot{\vartheta}$, and $\dot{\psi}$.

As two illustrative examples for the simplifcations arising from the early use of Hamilton's principle to treat the motion of rigid bodies we treat the standard-textbook cases of the force-free and the heavy symmetric top.

6 Force-free symmetric top

In this case the rigid body is fixed in its center of mass, and we have $\vec{s} = 0$. From (58) we immediately know that the total angular momentum is conserved, which is due to the full rotational invariance around the center of mass. Since in the space-fixed frame the Euler angle ψ parametrizes a rotation around the $\vec{e}_3$-direction, it is convenient to choose the body-fixed basis such that $\vec{J} = J\vec{e}_3$. The conservation of angular momentum then implies $J = \text{const.}$

Now the components of the angular momentum wrt. body-fixed coordinates is, using (56) and (67)

$$\underline{J}' = \begin{pmatrix} A\omega_1' \\ A\omega_2' \\ C\omega_3' \end{pmatrix} = \begin{pmatrix} A(\dot{\vartheta}\cos\varphi + \dot{\psi}\sin\vartheta\sin\varphi) \\ A(-\dot{\vartheta}\sin\varphi + \dot{\psi}\sin\vartheta\cos\varphi) \\ C(\dot{\varphi} + \dot{\psi}\cos\vartheta) \end{pmatrix}. \tag{73}$$

On the other hand using the rotation matrix $\hat{D}$ given in (61) yields

$$\underline{J}' = J\hat{D}^{\mathrm{T}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = J \begin{pmatrix} \sin\vartheta\sin\varphi \\ \sin\vartheta\cos\varphi \\ \cos\vartheta \end{pmatrix}. \tag{74}$$

Because φ is a cyclic coordinate, $p_\varphi = \text{const}$, and thus (70) implies $J_3' = p_\varphi = \text{const}$. Because of (74) we have $\vartheta = \vartheta_0 = \text{const}$. Since according to (71) then also $\dot{\psi} = \Omega_1 = \text{const}$ and consequently due to (70) $\dot{\varphi} = \Omega_2 = \text{const}$.

Using these consequences of the symmetry of the problem in (73) and compare to (74) we find $J = A\Omega_1 = C(\Omega_2 + \Omega_1\cos\vartheta_0)/\cos\vartheta_0$ and thus

$$\underline{J}' = \begin{pmatrix} A\Omega_1 \sin\vartheta_0 \sin(\Omega_2 t + \varphi_0) \\ A\Omega_1 \sin\vartheta_0 \cos(\Omega_2 t + \varphi_0) \\ C(\Omega_2 + \Omega_1\cos\vartheta_0) \end{pmatrix} = A\Omega_1 \begin{pmatrix} \sin\vartheta_0 \sin(\Omega_2 t + \varphi_0) \\ \sin\vartheta_0 \cos(\Omega_2 t + \varphi_0) \\ \cos\vartheta_0 \end{pmatrix}. \tag{75}$$

This implies that for a body-fixed observer the angular momentum precedes around the fixed figures axis $\vec{e}_3'$ with angular velocity Ω_2, while $J_3' = \text{const}$. The same holds for the momentaneous axis of rotation, given by

$$\underline{\omega}' = \hat{\Theta}'^{-1}\underline{J}' = \Omega_1 \begin{pmatrix} \sin\vartheta_0 \sin(\Omega_2 t + \varphi_0) \\ \sin\vartheta_0 \cos(\Omega_2 t + \varphi_0) \\ (A/C)\cos\vartheta_0 \end{pmatrix}. \tag{76}$$

We further note that $\underline{J}' - A\underline{\omega}' = (0, 0, (A - A^2/C)\Omega_1 \cos\vartheta_0)^{\mathrm{T}}$, which implies that the angular momentum $\vec{J}$, the momentary angular velocity $\vec{\omega}$, and the figure axis $\vec{n}_f = \vec{e}_3'$ are always in a plane.
To describe the experiments usually performed with spinning symmetric tops we also analyze the motion from the point of view of a space-fixed observer. The figure axis $\vec{n}_f = \vec{e}_3'$ has the space-fixed components

$$\underline{n}_f = \hat{D}\underline{n}_f' = \hat{D}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}\sin\vartheta_0 \sin(\Omega_1 t + \psi_0)\\ -\sin\vartheta_0 \cos(\Omega_1 t + \psi_0)\\ \cos\vartheta_0\end{pmatrix}, \tag{77}$$

i.e., it precesses around the constant angular momentum $\underline{J} = (0, 0, J)^{\mathrm{T}}$ with angular velocity Ω_1. Finally from (65) we obtain the angular velocity of the body as seen from the space-fixed observer as

$$\underline{\omega} = \begin{pmatrix}\Omega_2 \sin\vartheta_0 \sin(\Omega_1 t + \psi_0)\\ -\Omega_2 \sin\vartheta_0 \cos(\Omega_1 t + \psi_0)\\ \Omega_1 + \Omega_2 \cos\vartheta_0\end{pmatrix}. \tag{78}$$

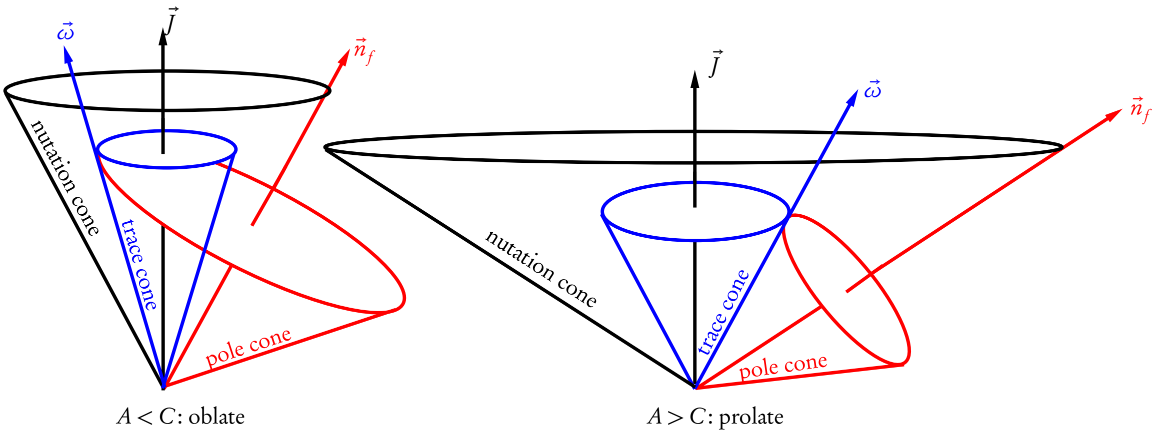
This means that also $\underline{\omega}$ precesses around the constant angular momentum also with angular velocity Ω_1.
Finally we investigate the angles between the various vectors. For that we introduce the variables

$$\omega_3' = \Omega_2 + \Omega_1 \cos\vartheta_0 = \omega_0, \quad \omega_\perp' = \sqrt{\omega_1'^2 + \omega_2'^2} = \Omega_1 \sin\vartheta_0 = \alpha_0. \tag{79}$$

Then we find

$$\begin{aligned}
\cos\angle(\vec{n}_f, \vec{\omega}) &= \frac{\vec{n}_f \cdot \vec{\omega}}{|\vec{\omega}|} = \frac{\underline{n}_f' \cdot \underline{\omega}_f'}{|\underline{\omega}|} = \frac{\omega_0}{\sqrt{\alpha_0^2 + \omega_0^2}},\\
\cos\angle(\vec{n}_f, \vec{J}) &= n_{f3} = \cos\vartheta_0 = \frac{\underline{n}_f' \cdot \underline{J}'}{J} = \frac{J_3'}{J} = \frac{\omega_0}{\sqrt{\omega_0^2 + (A/C)^2 \alpha_0^2}},\\
\cos\angle(\vec{\omega}, \vec{J}) &= \frac{\underline{J}' \cdot \underline{\omega}'}{J|\vec{\omega}|} = \frac{A\alpha_0^2 + C\omega_0^2}{\sqrt{(\alpha_0^2 + \omega_0^2)(A^2\alpha_0^2 + C^2\omega_0^2)}}.
\end{aligned} \tag{80}$$

For $A < C$ ("oblate symmetric top") $\cos\angle(\vec{n}_f, \vec{\omega}) < \cos\angle(\vec{n}_f, \vec{J})$, i.e., $\angle(\vec{n}_f, \vec{J}) < \angle(\vec{\omega}, \vec{J})$ and the other way for $A > C$ ("prolate symmetric top"). So we can depict the motion of the top for these two cases as



Seen from the space-fixed frame the angular momentum $\vec{J}$ is constant, and the angular velocity precesses with constant angular frequency Ω_1 around $\vec{J}$ and sweeps out the trace cone (blue). The figure axis also precesses around $\vec{J}$ with the same constant angular frequency Ω_1 and sweeps out the nutation cone (black). For an observer co-moving with the body both the angular momentum and $\vec{\omega}$ precess with the constant angular frequency Ω_2 around the fixed figure axis; $\vec{\omega}$ swipes out the pole cone (red). The three vectors $\vec{n}_f$, $\vec{\omega}$, and $\vec{J}$ are always in a plane, and the body-fixed pole cone, seen from the space-fixed frame, rolls without slipping on the space-fixed trace cone. For an oblate (prolate) top the trace cone is always located inside (outside) of the pole cone. The intersection line of these cones is always $\vec{\omega}$.

7 The heavy symmetric top

Now we investigate the case, where the symmetric rigid body is fixed at a point on its figure axis, that is not the center of mass. Then $\vec{s} = s\vec{e}_3'$, where $\vec{e}_3' = \vec{n}_f$ is the direction of the figure axis. We choose the space-fixed axis $\vec{e}_3$ such that $\vec{g} = -g\vec{e}_3$. Then

$$V = -M\vec{g}\cdot\vec{s} = Mgs\vec{e}_3\cdot\vec{e}_3' = Mgs\cos\vartheta. \tag{81}$$

Then the Lagrangian (69) reads

$$L = \frac{1}{2}\left[A(\dot{\vartheta}^2 + \dot{\psi}^2\sin\vartheta) + C(\dot{\varphi} + \dot{\psi}\cos\vartheta)^2\right] - Mgs\cos\vartheta. \tag{82}$$

Since ψ and φ are still cyclic and L is no explicitly time-dependent the system is integrable since we have three conserved quantities

$$p_\psi = \frac{\partial L}{\partial\dot{\psi}} = A\dot{\psi}\sin^2\vartheta + C(\dot{\varphi} + \dot{\psi}\cos\vartheta)\cos\vartheta \tag{83}$$

$$p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = C(\dot{\varphi} + \dot{\psi}\cos\vartheta), \tag{84}$$

$$E = T + V = \frac{1}{2}\left[A\left(\dot{\psi}^2\sin^2\vartheta + \dot{\vartheta}^2\right) + C\left(\dot{\varphi} + \dot{\psi}\cos\vartheta\right)^2\right] + Mgs\cos\vartheta. \tag{85}$$

These are the component of the angular-momentum component along the space-fixed axis $\vec{e}_3$ (due to invariance of the system under rotations around the direction of $\vec{g}$), the angular-momentum component along the body-fixed figure axis $\vec{e}_3'$ (due to the symmetry of the body's tensor of inertia around the $\vec{e}_3'$ axis) , and the energy (due to time-translation invariance).
Usin (83) and (84) to eliminate $\dot{\psi}$ and $\dot{\varphi}$ in (85) leads to

$$E = \frac{A}{2}\dot{\vartheta}^2 + \frac{(p_\psi - p_\varphi\cos\vartheta)^2}{2A\sin^2\vartheta} + \frac{p_\varphi^2}{2C} + Mgs\cos\vartheta. \tag{86}$$

As usual this admits the solution for the equation of motion for ϑ in implicit form. After substitution of $u = \cos\vartheta$ we get

$$t - t_0 = \pm\int_{u_0}^{u}\mathrm{d}u\frac{A}{\sqrt{2A(E - Mgsu)(1-u^2) - (A/C)p_\varphi^2(1-u^2) - (p_\psi - p_\varphi u)^2}}. \tag{87}$$

The solution for u leads to some elliptic function. In the following we restrict ourselves to a qualitative discussion of the motion.

The sign of the square root has to be chosen in the usual way: for physically consistent initial conditions the expression under the root must have two real roots $u_1, u_2 \in [-1, 1]$ between which it takes positive values. The figure axis thus oscillations wrt. the body-fixed axis $\vec{e}_3 \parallel \vec{g}$ between the corresponding values ϑ_1 and ϑ_2. In the here considered case of the heavy top this oscillation is called **nutation**. From (83) and (84) we get

$$\dot{\psi} = \frac{p_\psi - p_\varphi \cos\vartheta}{A\sin^2\vartheta}. \tag{88}$$

This means that the figure axis rotates also around $\vec{e}_3$ with an angular velocity oscillating between two values due to the nutation. This rotation is called **precession**. Let us label the roots of the argument under the root in (87) such that $u_1 = \cos\vartheta_1 < u_2 = \cos\vartheta_2$. If $p_\psi > p_\varphi u_2$ the angular velocity is always $\dot{\psi} > 0$, and the precession is always progressive in one direction. For $p_\psi = p_\varphi u_2$, the precession stops when the figure axis becomes steepest. Finally, if $p_\psi < p_\varphi u_2$, then $\dot{\psi}$ changes sign and the precession becomes retrograde for a while. This can be depicted by drawing the end of the figure-axis direction $\vec{n}_f = \vec{e}_3'$ on a sphere:

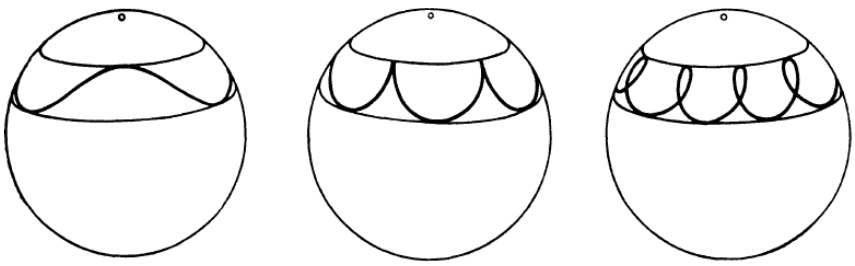


Figure 2: The trace of the end point of the figure-axis vector $\vec{n}_r = \vec{e}_3'$ for the three cases described in the text. Figure taken from [Wei63].

Depending on the sign of $\dot{\psi}$ this can be both a rotation in one direction (if $\dot{\psi}$ has only one sign for all times) or in different directions (if $\dot{\psi}$ changes sign).

A special case is $\vartheta = \text{const}$ but $\vartheta \notin \{0, \pi\}$. Then $\vartheta_1 = \vartheta_2$ corresponds to a root $u_1 = u_2$ of degree 2 of the argument of the square root in (87). Then according to (88) also $\dot{\psi} = \text{const}$, i.e., we have a case of **regular precession**. Then (84) also implies $\dot{\varphi} = 0$.

Pseudoregular precession

One typical situation usually demonstrated in experimental-physics lectures in mechanics is that the top is brought to fast rotation around the figure axis and then set on a needle with a point on the figure axis taking an angle ϑ_0 wrt. $\vec{e}_3$ without any further kick. Then we have the initial conditions

$$\vartheta(0) = \vartheta_0, \quad \psi(0) = \varphi(0) = 0, \quad \dot{\varphi}(0) = \omega_0, \quad \dot{\psi}(0) = \dot{\vartheta}(0) = 0. \tag{89}$$

We now assume that

$$\vartheta = \vartheta_0 + \epsilon \tag{90}$$

with $|\epsilon| \ll 1$. In the following we analyze the equations of motion to linear order in ϵ and its time derivatives. Of course, we then also have to discuss under which circumstances this approximation is valid at all times. From the initial conditions we find from (83-85)

$$p_\psi = C\omega_0 \cos\vartheta_0, \quad p_\varphi = C\omega_0, \quad E = \frac{p_\varphi^2}{2C} + Mgs\cos\vartheta_0. \tag{91}$$

Plugging the ansatz (90) in (86) and expand to order ϵ^2, we find

$$\frac{A}{2}\dot{\vartheta}^2 = \frac{A}{2}\dot{\epsilon}^2 = Mgs\sin\vartheta_0\epsilon + \frac{1}{2}\left(Mgs\cos\vartheta_0 - \frac{C^2\omega_0^2}{A}\right)\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{92}$$

Taking the time derivative of this equation then leads to the equation of motion

$$\ddot{\epsilon} = \Omega^2(a-\epsilon), \tag{93}$$

where we defined

$$\Omega^2 = \frac{C^2\omega_0^2}{A^2} - \frac{Mgs}{A}\cos\vartheta_0, \quad a = \frac{Mgs\sin\vartheta_0}{A\Omega^2}. \tag{94}$$

Obviously ϵ stays small only if $\Omega^2 > 0$ since then the general solution reads

$$\epsilon(t) = c_1\cos(\Omega t + c_2) + a. \tag{95}$$

With the initial conditions $\epsilon(0) = \dot{\epsilon}(0) = 0$ corresponding to the above described situation we can determine the integration constants c_1 and c_2 to get

$$\epsilon(t) = a[1-\cos(\Omega t)]. \tag{96}$$

The linear approximation is thus justified if $|a| \ll 1$, which according to (93) is fufilled if

$$Mgs \ll \frac{C^2\omega_0^2}{A}. \tag{97}$$

This means that the potential energy of the gravitational force is much smaller compared to the initial rotational energy.
Expanding (88) to linear order in ϵ, using (91 leads to

$$\dot{\psi} = \frac{C\omega_0}{A\sin\vartheta_0}\epsilon + \mathcal{O}(\epsilon^2). \tag{98}$$

With (95) we obtain by integration using again the initial conditions (89)

$$\psi(t) = \frac{C\omega_0 a}{A\sin\vartheta_0}\left[t - \frac{\sin(\Omega t)}{\Omega}\right]. \tag{99}$$

This means that the top reacts to the torque due to the gravitational force with the precession of the figure axis around $\vec{e}_3 \parallel \vec{g}$, i.e., it evades the action of the force in perpendicular direction. Though $\dot{\psi}$ is not constant its fast oscillations with angular frequency Ω are not that obvious, i.e., we have a **pseudoregular precession**.
In the same way we find with (84) by expansion to order ϵ

$$\dot{\varphi} = \omega_0 - \frac{C\omega_0\cot\vartheta_0}{A}\epsilon + \mathcal{O}(\epsilon^2) \tag{100}$$

with the solution

$$\varphi(t) = \omega_0 t - \frac{C\omega_0 a\cot\vartheta_0}{A}\left[t - \frac{\sin(\Omega t)}{\Omega}\right]. \tag{101}$$

A Continuum-mechanical proof of Euler's theorem

Though pretty much intuitively evident we present a proof of Euler's theorem according to which the motion of a rigid body is described as the composition of a translational motion of one arbitrary point relative to it and a rotation around this point. The rigid body has thus 6 degrees of freedom: a position vector of the arbitrary body-fixed reference point relative to the space-fixed reference frame and 3 Euler angles of the rotation between an arbitrary space-fixed Cartesian basis and an arbitrary body-fixed right-handed Cartesian basis, as used above to describe the kinematics of the rigid body.

To that end we describe the body by **material coordinates** $\vec{\xi} \in V$ which define the body in some arbitrary location. One can, e.g., choose the point $\vec{\xi}$ at the initial time $t = 0$. Then the body at later times is described by $\vec{x}(\vec{\xi}, t)$, i.e., the location of the point in the body, which has been at the place $\vec{\xi}$ at $t = 0$. A vector connecting two infinitesimally close points separated by the shift $\delta\vec{\xi}$ of the material coordinates is then simply given by

$$\delta x_j = \delta\xi_k \frac{\partial x_j}{\partial \xi_k}, \tag{102}$$

where we use the usual Euclidean Ricci calculus with only lower indices.
The **condition for rigidity** now is that

$$\delta\vec{x}^2 = \frac{\partial x_j}{\partial \xi_k}\frac{\partial x_j}{\partial \xi_l}\delta\xi_k\delta\xi_l = \text{const.} \tag{103}$$

Defining $\vec{v}(\vec{\xi}, t) = \partial_t \vec{x}(\vec{\xi}, t)$, this equation reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\vec{x}^2 = \left(\frac{\partial v_j}{\partial \xi_k}\frac{\partial x_j}{\partial \xi_l} + \frac{\partial v_j}{\partial \xi_l}\frac{\partial x_j}{\partial \xi_k}\right)\delta\xi_k\delta\xi_l \stackrel{!}{=} 0. \tag{104}$$

Since the $\delta\xi_k$ are arbitrary variations, the bracket must identically vanish. Now we express this equation in terms of Eulerian coordinates, i.e., using

$$\frac{\partial v_j}{\partial \xi_k} = \frac{\partial v_j}{\partial x_{k'}}\frac{\partial x_{k'}}{\partial \xi_k}, \tag{105}$$

we get

$$\frac{\partial v_j}{\partial x_{k'}}\left(\frac{\partial x_{k'}}{\partial \xi_k}\frac{\partial x_j}{\partial \xi^l} + \frac{\partial x_j}{\partial \xi_k}\frac{\partial x_{k'}}{\partial \xi^l}\right) = 0. \tag{106}$$

Now we exchange the summation variables j and k' for the 2nd term on the left-hand side, which yields

$$\left(\frac{\partial v_j}{\partial x_{k'}} + \frac{\partial v_{k'}}{\partial x_j}\right)\frac{\partial x_j}{\partial \xi_l}\frac{\partial x_{k'}}{\partial \xi_k} = 0. \tag{107}$$

Since the Jacobian matrix $J_{jl} = \partial x_j / \partial \xi_l$ of the transformation from Lagrange to Euler coordinates is by assumption invertible, this is equivalent to the equation

$$\sigma_{jk} = \frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j} = 0. \tag{108}$$

Writing $\partial_k = \partial/\partial x_k$ we get

$$\partial_l \sigma_{jk} = 0. \tag{109}$$

It is easy to show that from this

$$\partial_l \sigma_{jk} + \partial_j \sigma_{kl} - \partial_k \sigma_{lj} = 0 \Rightarrow \partial_j \partial_l v_k = 0. \tag{110}$$

Now we use an arbitrary fixed material point with material coordinates $\vec{\xi}_0$ within the body, $\vec{x}_0(t) = \vec{x}(\vec{\xi}_0, t)$ and define $\vec{v}_0(t) = \partial_t \vec{x}_0(t)$. Then the general solution of (109) can be written as

$$v_k = \omega_{jk}(t)[x_j - x_{0j}(t)] + v_{0k}(t). \tag{111}$$

With (108) we further find the constraint

$$\omega_{jk}(t) = -\omega_{kj}(t) \Rightarrow \omega_{jk}(t) = \epsilon_{ijk}\omega_i(t), \tag{112}$$

where we have used that we can write any antisymmetric matrix as the "Hodge dual" of an axial vector. Then (111) reads

$$\vec{v}(t,\vec{x}) = \vec{v}_0(t) + \partial_t \vec{y}(\vec{\xi}, t) = \vec{\omega}(t) \times \vec{y}(\vec{\xi}, t) + \vec{v}_0(t), \tag{113}$$

where $\vec{y}(\vec{\xi}, t) = \vec{x}(\vec{\xi}, t) - \vec{x}_0(t)$. Then we find

$$\vec{y}(\vec{\xi}, t) = \hat{D}(t)\vec{y}(\vec{\xi}, 0), \tag{114}$$

where $\hat{D}(t) \in \mathrm{SO}(3)$. This proves the above heuristic argument about the three translational and three rotational degrees of freedom that describe the motion of a non-relativistic rigid body.

References

[Gre03] W. Greiner, Klassische Mechanik 1, Verlag Harri Deutsch, Frankfurt/Main, 7 ed. (2003).

[LL60] L. D. Landau, E. M. Lifshitz, Course of Theoretical Physics, Vol. 1, Mechanics, Pergamon Press, Oxford, London, Paris (1960).

[Som94] A. Sommerfeld, Vorlesungen über Theoretische Physik I, Mechanik, Verlag Harri Deutsch, Frankfurt/M. (1994).

[Wei63] W. Weizel, Lehrbuch der Theoretischen Physik Bd. 1, Springer Verlag, Berlin, Göttingen, Heidelberg, 3 ed. (1963).
`https://doi.org/10.1007/978-3-642-87337-9`