Relativistic treatment of the DC conducting coaxial cable

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1 Introduction

The direct-current-conducting infinitely long wire is often discussed in the context of relativistic electrodynamics. It is of course a completely academic discussion since for the typical household currents the drift velocity of the electrons in the wire, making up the conduction current, is tiny (of the order $\mathcal{O}(1\,\text{mm/s})$!). Nevertheless it is unfortunately only quite confusingly discussed in the literature. So here is my attempt for a more consistent description using a naive classical model of a metal as consisting of a continuum of effectively positive bound charges (consisting of atoms and the bound electrons making up the lattice forming the metal, in the following called "ions") and negative conduction electrons, treated as a freely moving fluid subject to some friction when moving against the positive charged rigid background.

The confusion starts with the fact that usually it is not carefully discussed in which reference frame the wire is uncharged. It is *not* the rest frame of the wire (i.e., the rest frame of the ions) but the rest frame of the conduction electrons [Pet85]. The qualitative argument is simple: We consider a straight wire with a constant current. In the rest frame of the conduction electrons there is a current due to the moving positively charged background and a corresponding magnetic field. The charges within the positive background are however bound and can be considered not to move relative to each other due to the electromagnetic field. Thus in this reference frame the charge density vanishes everywhere within the wire as if there were no current at all since the freely movable conduction electrons are at rest, and there is thus no net force acting on them and thus in this reference frame no charge separation occurs.

Since the charge density (times c) and the current density form a four-vector, consequently in the rest frame of the wire (i.e., the rest frame of the ions) there must be a non-vanishing charge density within the wire due to the Lorentz-transformation properties of vector components. This is also easily explained dynamically: In this reference frame the conduction electrons move along the wire with constant velocity, and thus a magnetic field is present, which causes a radial force on the conduction electrons, which consequently arrange such that an electric field is built up which exactly compensates this magnetic force. This is, of course, nothing else than the **"self-induced" Hall effect**.

This is taken into account automatically when the correct relativistic version of **Ohm's Law** is considered, which is sometimes approximated with the simple non-relativistic form, consequently leading to non-covariant approximations of the fields.

In contradistinction to an earlier version of this manuscript, the single current-conducting wire does not allow for a consistent description, which is understandable since one needs a closed circuit to have a direct current. So in the following we assume a very long coaxial cable consisting of a cylindrical straight wire of radius a_1 along the 3-axis of our Cartesian coordinate system and a parallel cylindrical shell of inner radius a_2 and outer radius a_3. We shall treat this problem entirely in the rest frame of the wire. The only necessary relativistic correction compared to the standard treatment [Som52] is the use of the correct relativistic formulation of Ohm's Law, which we derive using the classical Drude model of electric conduction.

The notation is as follows: Four-vectors are written as $\underline{x} = (x^0, \vec{x})$, the Minkowski fundamental form is given by $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$. Four-velocities are meant in dimensionless form, i.e., $\underline{u} = \gamma(1, \vec{\beta})$ with $\beta = \vec{v}/c$ and $\gamma = 1/\sqrt{1-\vec{\beta}^2}$.

2 Relativistic description of Ohm's Law

First we derive the correct relativistic version of Ohm's Law. To that end we consider the motion of an electron with charge $q_{\mathrm{e}} = -e$ in the electromagnetic field within the wire. In addition to the Lorentz force, there is also a friction force. We start to describe the motion of the electron in the manifestly covariant way with the four-velocity $\underline{u} = \gamma(1, \vec{\beta}) = \dot{x}/c$ with $\vec{\beta} = \vec{v}/c$, where $\dot{x} = \mathrm{d}x/\mathrm{d}\tau$. The equation of motion then reads

$$mc\,\mathrm{d}_\tau u^\mu = -eF^{\mu\nu}u_\nu + K^\mu_{\text{fric}}. \tag{1}$$

The Faraday tensor $F^{\mu\nu}$ is described in detail in [Hee19].

The relativistic version of the friction force, which we assume to be proportional to the velocity of the particle (Stokes friction) can be written in a manifestly covariant way by introducing the four-velocity (U^μ) of the medium (which in our case by definition is the four-velocity of the wire). Then we can write for the friction force on a conduction electron with four-velocity, u, and four-momentum $p = mcu$,

$$K^\mu_{\text{fric}} = \alpha(U^\mu u^\nu - U^\nu u^\mu)p_\nu = \alpha[mcU^\mu - (U \cdot u)p^\mu], \tag{2}$$

where α is the friction coefficient, which is obviously a Lorentz-scalar quantity.

For a direct current, i.e., for a time-independent j^μ we have $\mathrm{d}_\tau u = 0$. This implies $K^\mu_{\text{fric}} = eF^{\mu\nu}u_\nu$. From this we get the four-current of the conduction electrons

$$j^\mu_- = -n_{\text{cond}}ecu^\mu = \frac{\sigma}{U \cdot u}F^{\mu\nu}u_\nu - \frac{en_{\text{cond}}c}{U \cdot u}U^\mu, \tag{3}$$

where n_{cond} is the number density of the electrons as measured in their (local) rest frame (a Lorentz scalar) and the electric conductivity,

$$\sigma = \frac{n_{\text{cond}}e^2}{\alpha m}, \tag{4}$$

which also is a Lorentz scalar. The total four-current density within the wire reads

$$j^\mu_{\text{wire}} = \frac{\sigma}{U \cdot u}F^{\mu\nu}u_\nu + ec\left(n_+ - \frac{n_{\text{cond}}}{U \cdot u}\right)U^\mu, \tag{5}$$

where n_+ is the number density of positive ions, as defined in their (local) rest frame. In the following we work in the rest frame of the wire $(U^\mu) = (1,0,0,0)$, where the correct relativistic form of Ohm's Law is given by the spatial components of (5),

$$\vec{j} = \sigma(\vec{E} + \vec{\beta} \times \vec{B}). \tag{6}$$

We note that in this frame $\rho_{\text{wire}} = e(n_+ - n_{\text{cond}}\gamma) = \rho_+ + \rho_{\text{cond}}$ with $\rho_+ = en_+$ and $\rho_{\text{cond}} = -e\gamma n_{\text{cond}}$.

3 Covariant magnetostatics

Now we can write down the relativistic magnetostatic Maxwell's equations, expressed in (1+3)-notation in the restframe of the wire as

$$\vec{\nabla} \times \vec{E} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \tag{7}$$

$$\vec{\nabla} \cdot \vec{E} = \rho_{\text{wire}}, \tag{8}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \vec{j}_{\text{cond}} = \rho_{\text{cond}} \vec{\beta} \tag{9}$$

$$\vec{j}_{\text{cond}} = \rho_{\text{cond}} \vec{v} = \sigma(\vec{E} + \vec{\beta} \times \vec{B}). \tag{10}$$

For our DC-carrying coaxial cable we make the ansatz that within the wires $\rho_{\text{cond}} = \text{const}$ as well es $\vec{j}_{\text{cond}} = \text{const}$ and thus $\vec{\beta} = \text{const}$. We work in cylinder coordinates (R, φ, z) and define the regions

- I interior of the inner wire, $0 \leq R \leq a_1$,
- II free region between the conductors, $a_1 < R < a_2$,
- III the outer conductor $a_2 \leq R \leq a_3$,
- IV the region outside the cable, $R > a_3$.

We assume that there is a total current I running along the conductors. We do not discuss the voltage source at one and (which we push to $z \to -\infty$) nor some closure of the circuit by a connection of the two conductors via a resistor at the other end at $z \to +\infty$, i.e., we simply make the ansatz

$$\vec{j}^{(\text{I})} = \frac{I}{\pi a_1^2} \vec{e}_3, \quad \vec{j}^{(\text{II})} = -\frac{I}{\pi(a_3^2 - a_2^2)} \vec{e}_3, \quad \vec{j}^{(\text{III})} = \vec{j}^{(\text{IV})} = 0 \tag{11}$$

with $I > 0$. We note that this ansatz guarantees global charge conservation since integration over a plane, P perpendicular to the 3-axis gives

$$\int_P \mathrm{d}^2 \vec{f} \cdot \vec{j} = j^{(\text{I})} \pi a_1^2 + j^{(\text{II})} \pi (a_3^2 - a_2^2) = 0. \tag{12}$$

The homogeneous Maxwell equations (7) are solved by introducing the scalar and vector potentials,

$$\vec{E} = -\vec{\nabla}\Phi, \quad \vec{B} = \vec{\nabla} \times \vec{A}, \tag{13}$$

and due to gauge invariance we can impose the Coulomb-gauge constraint on the vector potential,

$$\vec{\nabla} \cdot \vec{A} = 0. \tag{14}$$

3.1 The magnetic field

Obviously the equations for the magnetic field completely decouple from those of the electric field. For the latter we also need the magnetic field. So we solve for the magnetic field first. Plugging (13) into (9) we get

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\Delta \vec{A} = \frac{1}{c} \vec{j}_{\text{cond}}. \tag{15}$$

Since $\vec{j}_{\text{cond}} = j_{\text{cond}} \vec{e}_z$ due to the cylindrical symmetry the ansatz

$$\vec{A} = A(R)\vec{e}_z \tag{16}$$

seems plausible. With this ansatz (14) is automatically fulfilled, and we get, using the formulae for the curl in cylindrical coordinates

$$\vec{B} = \vec{\nabla} \times \vec{A} = -A'(R)\vec{e}_\varphi = B\vec{e}_\varphi \tag{17}$$

and

$$\vec{\nabla} \times \vec{B} = -\vec{e}_z \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\left[RA'(R)\right]\vec{e}_z = -\Delta\vec{A} = \frac{j}{c}\vec{e}_z \tag{18}$$

With $j = \text{const}$ the general solution of this equation reads

$$A(R) = -\frac{j}{4c}R^2 + C_1 \ln(R/R_0) + C_2. \tag{19}$$

Here R_0 can be chosen arbitrarily. From (17) we find

$$B = -A' = \frac{j}{2c}R - \frac{C_1}{R}. \tag{20}$$

In region I we must have obviously $j = j^{(\mathrm{I})} = I/a_1^2$ and $C_1^{(\mathrm{I})} = 0$, because there are no singularities along the cylinder axis, $R = 0$. So we have

$$B^{(\mathrm{I})} = \frac{I}{2\pi c a_1^2}R. \tag{21}$$

In region II $j = 0$. Since there are no surface-current densities along the cylinders, the $\vec{B}$-field is continuous everywhere and thus $B^{(\mathrm{II})}(a_1) = C_1^{(\mathrm{II})}/a_1 = B^{(\mathrm{I})}(a_1) = I/(2\pi a_1 c)$ leads to

$$B^{(\mathrm{II})} = \frac{I}{2\pi c R}. \tag{22}$$

In region III $j = j^{(\mathrm{III})} = -I/[\pi(a_3^2 - a_1^2)]$ and thus

$$B^{(\mathrm{III})} = \frac{-I}{2\pi c(a_3^2 - a_1^2)}R - \frac{C_1^{(\mathrm{III})}}{R}. \tag{23}$$

Continuity at $R = a_2$ determines $C_1^{(\mathrm{III})}$ and

$$B^{(\mathrm{III})} = \frac{I}{2\pi c(a_3^2 - a_2^2)}\left(\frac{a_3^2}{R} - R\right). \tag{24}$$

In region IV $j = j^{(\mathrm{IV})} = 0$ and the continuity argument finally yields

$$B^{(\mathrm{IV})} = 0. \tag{25}$$

3.2 The electric field

To find the electric field, we first take the divergence of Ohm's Law (10) making use of (8) and (9)

$$\vec{\nabla}\cdot\vec{j}=0=\sigma\vec{\nabla}\cdot(\vec{E}+\vec{\beta}\times\vec{B})=\sigma\left(\rho_{\text{wire}}-\frac{1}{c}\vec{\beta}\cdot\vec{j}\right)=\sigma\left(\rho_{\text{wire}}-\rho_{\text{cond}}\beta^2\right). \tag{26}$$

This implies that

$$\rho_{\text{wire}}=\rho_{\text{cond}}\beta^2. \tag{27}$$

This shows that the conductors carry a charge density of the same sign as the particles responsible for the current, i.e., in our case of a usual metal the conduction electrons, i.e., $\rho_{\text{wire}}<0$.

As is clear from our model for electric conductivity this charging of the wires is due to the self-induced Hall effect, i.e., due to the magnetic field the magnetic Lorentz force drags the conduction electrons inside the wire, which causes an electric field counter acting this force.

Now we can determine the electric field in the conductors, i.e., in regions I and III from Ohm's Law (10) only. In both regions I and III

$$\vec{\beta}\times\vec{B}=\beta B\vec{e}_z\times\vec{e}_\varphi=-\beta B\vec{e}_R. \tag{28}$$

Because $\vec{j}=j\vec{e}_z$ we have

$$E_R=\beta B,\quad E_z=\frac{j}{\sigma} \tag{29}$$

It is clear that E_R adjusts such that the radial component of the net force on the conduction electrons vanishes, and E_z is responsible for the flow of the current along the wire.

In region I we have from (21)

$$E_R^{(\mathrm{I})}=\frac{I\beta^{(\mathrm{I})}}{2\pi a_1^2 c}R<0,\quad E_z^{(\mathrm{I})}=\frac{I}{\pi a_1^2\sigma}. \tag{30}$$

In the same way we obtain the electric field in region III, using (24)

$$E_R^{(\mathrm{III})}=\frac{I\beta^{(\mathrm{III})}}{2\pi c(a_3^2-a_2^2)}\left(\frac{a_3^2}{R}-R\right)>0,\quad E_z^{(\mathrm{III})}=-\frac{I}{\pi(a_3^2-a_2^2)\sigma}. \tag{31}$$

In the free regions II and IV we have $\rho_{\text{wire}}=0$ and $j=0$ as well as $\sigma=0$. In these regions we use (13) in (8), leading to the Laplace equation for the scalar potential,

$$\Delta\Phi=0. \tag{32}$$

In addition we need the continuity of the electric-field components tangential to the surfaces of the wire. As it turns out, the separation ansatz

$$\Phi=f(R)G(z) \tag{33}$$

is sufficient to fulfill all conditions. With the Laplace operator in cylinder coordinates, we get

$$\frac{1}{R}[Rf'(R)]'g(z)+f(R)g''(z)=0. \tag{34}$$

Obviously, since we need $f(R) \neq 0$ and $g(z) \neq 0$, we can fulfill this with g linear in z,

$$g''(z) = 0 \Rightarrow g(z) = G_1 z + G_2, \tag{35}$$

$$[Rf'(R)]' = 0 \Rightarrow f(R) = F_1 \ln\left(\frac{R}{R_0}\right) + F_2 \tag{36}$$

with integration constants, G_1, G_2, F_1, and F_2. Obviously we can set $G_1 = 1$ without loss of generality, because we can lump this constant into F_1 and F_2.
This implies

$$\vec{E} = -f'(R)g(z)\vec{e}_R - f(R)g'(z)\vec{e}_z = -\frac{F_1}{R}(z + G_2)\vec{e}_R - \left[F_1 \ln\left(\frac{R}{R_0}\right) + F_2\right]\vec{e}_z. \tag{37}$$

In region II it's most convenient to set $R_0 = a_1$. Then we can determine the constants F_1 and F_2 from the continuity of $E_z^{(\text{II})}$ at $R = a_1$ and $R = a_2$, i.e.,

$$E_z^{(\text{I})}(a_1) = \frac{I}{\pi a_1^2 \sigma} = E_z^{(\text{II})}(a_1) = -F_2 \Rightarrow F_2 = -\frac{I}{\pi a_1^2 \sigma}, \tag{38}$$

$$E_z^{(\text{II})}(a_2) = -\frac{I}{\pi(a_3^2 - a_2^2)\sigma} = E_z^{(\text{III})}(a_2) = -F_1 \ln\left(\frac{a_2}{a_1}\right) - F_2 \tag{39}$$

The solution finally reads

$$E_z^{(\text{II})} = \frac{I}{\pi a_1^2 \sigma} - \frac{I}{\ln(a_2/a_1)\sigma}\left(\frac{1}{a_1^2} + \frac{1}{a_3^2 - a_2^2}\right)\ln\left(\frac{R}{a_1}\right). \tag{40}$$

For the radial component from (37) we get

$$E_R^{(\text{II})} = -\frac{Iz}{\ln(a_2/a_1)\sigma R}\left(\frac{1}{a_1^2} + \frac{1}{a_3^2 - a_2^2}\right). \tag{41}$$

We have arbitrarily assumed that $G_2 = 0$, i.e., $E_R^{(\text{II})} = 0$ at $z = 0$[1].

The same arguments also apply to region IV. There we must impose the continuity condition on $E_z^{(\text{IV})}$ at $R = a_3$ and make sure that for $R \to \infty$ the field must stay finite, which implies $F_1 = 0$ in (37). The result is

$$E_z^{(\text{IV})} = -\frac{I}{\pi(a_3^2 - a_2^2)\sigma}, \quad E_R^{(\text{IV})} = 0. \tag{42}$$

We note that from (27) the non-vanishing components of the four-currents in the conductors are

$$\begin{pmatrix} j^0 \\ j^3 \end{pmatrix} = \rho_{\text{cond}} \beta c \begin{pmatrix} \beta \\ 1 \end{pmatrix}. \tag{43}$$

Lorentz-boosting to the rest-frame of conduction electrons gives

$$\begin{pmatrix} j'^0 \\ j'^3 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} j^0 \\ j^3 \end{pmatrix} = \frac{\beta c}{\gamma} \rho_{\text{cond}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{44}$$

This shows that indeed in the rest frame of the conduction electrons the charge density within the wires vanishes. One should also note that $\rho'_+ = -\rho_{\text{cond}}/\gamma$ is the charge density of the ions in the rest frame of the electrons. In this frame the velocity of the ions is $-\beta c \vec{e}_3$. So the current density in the rest frame of the conduction electrons within the wires is due to the motion of the ions, as it must be.

[1] The full physical determination of G_1 and G_2 will become clear in Sec. 7, when we approximately treat a coaxial cable of finite length with an ideal voltage source at one end at $z = 0$ and a resistor at the other end at $z = l$.

4 Surface charges

In this Section we evaluate the surface charges on the inner conductor at $R = a_1$ and the outer conductor at $R = a_2$. It is given by using an infinitesimal Gaussian pillbox around the corresponding surface. We find

$$\Sigma^{(\mathrm{I})} = E_R^{(\mathrm{II})}(a_1) - E_R^{(\mathrm{I})}(a_1) = -\frac{I\beta^{(\mathrm{I})}}{2\pi a_1 c} - \frac{Iz}{\ln(a_2/a_1)a_1\sigma}\left(\frac{1}{a_1^2} + \frac{1}{a_3^2 - a_2^2}\right), \tag{45}$$

$$\Sigma^{(\mathrm{III})} = E_R^{(\mathrm{IV})}(a_2) - E_R^{(\mathrm{II})}(a_2) = \frac{I\beta^{(\mathrm{III})}}{2\pi a_2 c} + \frac{Iz}{\ln(a_2/a_1)a_2\sigma}\left(\frac{1}{a_1^2} + \frac{1}{a_3^2 - a_2^2}\right), \tag{46}$$

There is no surface charge at $R = a_4$.

5 Charge neutrality

Finally we note that overall the currents are charge neutral. To see this we calculate the line-charge density of the entire coaxial cable. The charge per length in each of the conductors consists of both the bulk charge inside the conductors and the surface charges (45) and (46). With (8) we can express the charge densities inside the conductors, using (30) and (31)

$$\rho^{(\mathrm{I})} = \vec{\nabla} \cdot \vec{E}^{(\mathrm{I})} = \frac{1}{R}\left(RE_R^{(\mathrm{I})}\right)' = \frac{\beta^{(\mathrm{I})} I}{\pi a_1^2 c}, \quad \rho^{(\mathrm{III})} = \vec{\nabla} \cdot \vec{E}^{(\mathrm{III})} = \frac{1}{R}\left(RE_R^{(\mathrm{III})}\right)' = -\frac{\beta^{(\mathrm{I})} I}{\pi(a_3^2 - a_2^2)c}. \tag{47}$$

For the total line charge we then indeed find, using (45-47)

$$\lambda(z) = \rho^{(\mathrm{I})}\pi a_1^2 + \rho^{(\mathrm{III})}\pi(a_3^2 - a_2^2) + 2\pi a_1 \Sigma^{(\mathrm{I})} + 2\pi a_2 \Sigma^{(\mathrm{III})} = 0. \tag{48}$$

6 The "single-wire limit"

The often discussed somewhat artificial case of a single infinite cylindrical wire carrying a direct current can be interpreted as the limit, $a_2 \to \infty$. In this limit one has of course only region I (the interior of the wire) and II (the free space outside). Then the fields are given by (21), (22), (30), (40), and (41) in the said limit,

$$\begin{aligned} B^{(\mathrm{I})} &= \frac{I}{2\pi c a_1^2} R, \\ E_R^{(\mathrm{I})} &= \frac{I\beta^{(\mathrm{I})}}{2\pi a_1^2 c} R, \\ E_z^{(\mathrm{I})} &= \frac{I}{\pi a_1^2 \sigma}, \\ E_R^{(\mathrm{II})} &= 0, \quad E_z^{(\mathrm{II})} = \frac{I}{\pi a_1^2 \sigma}. \end{aligned} \tag{49}$$

The charge density in the wire and the surface-charge density on the boundary of the wire are given by (47) and (45) respectively

$$\rho^{(\mathrm{I})} = \frac{\beta^{(\mathrm{I})} I}{\pi a_1^2 c}, \quad \Sigma^{(\mathrm{I})} = -\frac{I\beta^{(\mathrm{I})}}{2\pi a_1 c}. \tag{50}$$

This solution has been found in [Gab93].

7 Approximate solution for a coaxial cable of finite length

TBD

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