(Not) Against photons

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1 Introduction

In this Insights article, I'd like to address again the issue of the abuse of the word **photon**, which often leads to debates in the forums and was already discussed in my previous Insights article for the special case of abusing the notion of photons to explain the photoelectric effect like Einstein in one of his famous articles of 1905.

The more general abuse of the word "photon" when discussing classical properties of light is due to one of the sins of physics didactics, which is in my opinion even a mortal sin, because it exposes young students (often already highschool students) with old-fashioned concepts which are outdated for more than 90 years, because then modern quantum theory has been discovered. In addition, the idea of a "light particle" in a naive sense is particularly misleading, because according to our contemporary modern understanding (which reaches as far back as 1927, when Dirac introduced the notion of the quantized electromagnetic field, which is the only way to make sense of quantum effects related to electromagnetism) a photon does not even allow for the definition of a position operator, and the classical limit of quantum electrodynamics (QED), as far as radiation is concerned, does not admit a naive classical-particle limit but rather a classical-field limit. So, if you do not need the quantized electromagnetic field to explain a phenomenon but can do with classical electrodynamics and Maxwell's equations, you better use the latter. If you really need photons the only securely correct way is to use QED anyway, and the good news is that QED is not as difficult as one might think.

The title of the article is due to a famous article by W. E. Lamb [Lam95] with the title "Anti-Photon". I recommend to read this paper, because it gives a brief but complete summary about the historical development of the theory of electromagnetic radiation (light) and the ideas about photons.

On the other hand, in my opinion, Lamb's advice not to use the word "photon" anymore, is a bit unrealistic and also unjustified. One should, however use it in an appropriate way, clearly defining what is implied by its notion according to modern relativistic quantum field theory.

In the following, I try to do this as simple as possible but not simpler, and unfortunately the issue is not as simple as non-relativistic quantum mechanics and in some details not even as simple as the relativistic theory of massive particles, when it comes to the description of interactions between particles or condensed matter and radiation. Only the free radiation field is really simple. Fortunately, it is the most important part in defining what a photon really is in the modern sense.

Another dilemma in such a short article is the choice of topics, because in the last about 30 years, photons are used for the most astonishing experiments concerning the basic principles of quantum theory. In principle photons occur in two contexts:

(a) in high-energy particle and nuclear physics, where one investigates the production of photons either in collisions of (a few) elementary particles (usually either in e^+e^- collisions, pp collisions, or in relativistic heavy-ion collisions, where photons are one of the very important probes for the hot and dense medium made up of quite many strongly interacting particles (quarks and gluons in the scalled quark-gluon plasma phase in the early as well as hadrons in the later stages of the evolution of the hot and dense "fireball" created in the collision).

(b) in quantum optics, where one uses usual optical elements like beam splitters, mirrors, lenses, gratings, etc. to investigate radiation consisting of a few (or even a single) photon. Here, one applies a semiclassical theory, where the optical elements are usually treated effectively as in classical electrodynamics but the radiation field must be treated as a quantum field. This is justified, because the dispersion theory of light in dielectrics or metals in terms of quantum theory in the here applicable linear-response approximation boils down to the introduction of effective macroscopic parameters like the (usually complex) index of refraction and/or electric conductivity.

    On the other hand, also strong coherent fields of radiation due to various kinds of **lasers** are available for more than 50 years now, and this has brought also non-linear optics into the focus. For our topic the most important development are highly efficient sources of **entangled photon pairs** in birefringent crystals.

The rest of the article is structured as follows:
First (Sect. 3) I give a very short review of the Maxwell equations in a vacuum, which is then used in Sect. 4 to canonically quantize the free radiation field in the fully gauge-fixed formalism ("radiation gauge"). Then in Sect. 6 I discuss the classical limit in the sense of states of the quantized electromagnetic field that are well described by classical electromagnetism. As the first application of the formalism we derive the Planck spectrum for black-body radiation from QED, which is the modern version of the historical starting point of both quantum theory in general and the quantum theory of radiation (Sect. 5). Then I review the theory for the two most simple proofs that single photons in the modern sense really exist, namely a beam-splitter experiment with single photons (Sect. 7) and quantum beats (Sect. 10). I hope with these very simple applications, I can convince the users of the Physics Forum that no inconsistent old-fashioned concepts like "wave-particle dualism" or even a naive particle picture for photons is necessary. I hope that one day textbook writers refrain from starting quantum-mechanics textbooks with a historical overview overemphasizing these very confusing outdated ideas, which are not only quantitatively but also qualitatively wrong, and finally these wrong ideas finally die out. It's high time for that, more than 90 years after the discovery of modern quantum theory.

2 Heuristics

TBD

3 Classical Maxwell theory in a vacuum

First we have to treat a minimum of classical theory of electromagnetic radition. In the following I use Heaviside-Lorentz units with $\hbar = c = 1$.
The Maxwell equations and with charge and current densities in this system read

$$\vec{\nabla} \times \vec{E} + \partial_t \vec{B} = 0, \quad \vec{\nabla} \cdot \vec{B} = 0, \tag{1}$$

$$\vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \vec{j}, \quad \vec{\nabla} \cdot \vec{E} = \varrho. \tag{2}$$

Here $(\vec{E}, \vec{B})$ are the electric and magnetic components of the electromagnetic field with respect to an arbitrary inertial reference frame and $(\rho, \vec{j})$ the charge and current density, which are the sources of the electromagnetic field.
The first two equations (1) are the **inhomogeneous Maxwell equation**, i.e., they are constraints on the electromagnetic field, and these can be implemented by introducing a scalar and a vector potential. Starting from the 2^{nd} equation, we conclude that according to Helmholtz's fundamental theorem of vector calculus there must exist a **vector potential** for the magnetic field,

$$\vec{B} = \vec{\nabla} \times \vec{A}. \tag{3}$$

The first equation in (1) now reads

$$\vec{\nabla} \times (\vec{E} + \partial_t \vec{A}) = 0, \tag{4}$$

and, again using Helmholtz's theorem, we can express the vector field under the curl as the gradient of a **scalar potential**

$$\vec{E} + \partial_t \vec{A} = -\vec{\nabla}\Phi \Rightarrow \vec{E} = -\partial_t \vec{A} - \vec{\nabla}\Phi. \tag{5}$$

Writing $(\vec{E},\vec{B})$ in terms of the potentials Φ and $\vec{A}$, we can forget about (1) and just solve for the potentials according to (2). Plugging in (3) and (5) for the electromagnetic field into (2), we find

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{A})+\partial_t^2\vec{A}+\partial_t\vec{\nabla}\Phi=\vec{j}, \tag{6}$$

$$-\vec{\nabla}(\vec{\nabla}\Phi+\partial_t\vec{A})=\varrho. \tag{7}$$

Restricting ourselves to Cartesian coordinates, we can rewrite (6) as

$$\Box\vec{A}+\vec{\nabla}(\partial_t\Phi+\vec{\nabla}\cdot\vec{A})=\vec{j} \tag{8}$$

with the **d'Alembert opertor**

$$\Box=\partial_t^2-\vec{\nabla}\cdot\vec{\nabla}=\partial_t^2-\Delta. \tag{9}$$

Now the potentials are not unique for a given physical situation, which is completely determined by the fields $(\vec{E},\vec{B})$ and sources $(\rho,\vec{j})$. Now, given the fields, instead of $\vec{A}$ in (3) we can use

$$\vec{A}'=\vec{A}-\vec{\nabla}\chi \;\Rightarrow\; \vec{\nabla}\times\vec{A}'=\vec{\nabla}\times\vec{A}=\vec{B} \tag{10}$$

with an arbitrary scalar field χ. To fulfill also (5) we must also change the scalar potential to another one Φ' such that

$$\vec{E}=-\partial_t\vec{A}'-\vec{\nabla}\Phi'=-\partial_t(\vec{A}-\vec{\nabla}\chi)-\vec{\nabla}\Phi'\stackrel{!}{=}-\partial_t\vec{A}-\vec{\nabla}\Phi. \tag{11}$$

Thus we can set

$$\Phi'=\Phi+\partial_t\chi. \tag{12}$$

The transformation from one set of potentials $(\Phi,\vec{A})$ to a new one $(\Phi',\vec{A}')$ according to Eqs. (10) and (11) is called a **gauge transformation**.

So the solution of the equations (6) and (7) for the potentials is determined only up to a gauge transformation, i.e., the scalar field χ via (10) and (12) is arbitrary and indetermined by any equation. Since its choice is completely arbitrary and without any effect to physically observable quantities, We can simplify our task to find these solution by introducing an arbitrary **gauge constraint** on the potentials. Looking at (8) suggests that a convenient choice is the **Lorenz-gauge condition**

$$\partial_t\Phi+\vec{\nabla}\cdot\vec{A}=0, \tag{13}$$

which decouples the Cartesian components of the vector potential, leading to separate **wave equations** for each component

$$\Box\vec{A}=\vec{j}, \tag{14}$$

and using (13) to substitute $\vec{\nabla}\cdot\vec{A}=-\partial_t\Phi$ one also obtains a wave equation for the scalar potential

$$\Box\Phi=\varrho. \tag{15}$$

This choice of the Lorenz gauge is convenient, because it becomes manifestly Lorentz covariant in the relativistic formulation of electrodynamics. For our purposes of quantizing the free radiation field it has the disadvantage of not completely fixing the gauge, because obviously if we have found any potentials $(\Phi,\vec{A})$, fulfilling the Lorenz-gauge condition (13) also the gauge transformed fields

$$\Phi'=\Phi+\partial_t\chi, \quad \vec{A}'=\vec{A}-\vec{\nabla}\chi \tag{16}$$

fullfil this condition, provided that

$$\Box\chi = 0, \tag{17}$$

i.e., under the restriction that the gauge field χ solves the homogeneous wave equation.

For free fields, i.e., when setting $\varrho = 0$ and $\vec{j} = 0$ we can easily fix the gauge completely by demanding the additional constraint

$$\Phi = 0, \tag{18}$$

because, if we have potentials $(\Phi, \vec{A})$ fulfilling the Lorenz-gauge condition (13) and the wave-equations of motion (14) and (15) we can define a new set via the gauge transformation with the gauge field

$$\chi = -\int \mathrm{d}t \Phi(t, \vec{x}), \tag{19}$$

because then

$$\Box\chi = -\partial_t \Phi + \int \mathrm{d}t \Delta\Phi, \tag{20}$$

but because of $\varrho = 0$ we have $\Box\Phi = 0$ and thus $\Box\chi = 0$. So we finally have *only a two-component vector potential* left, because we must fulfill the Lorenz-gauge condition (13) and the additional gauge constraint (18). These constraints we can write as the **radiation-gauge constraints**

$$\Phi = 0, \quad \vec{\nabla} \cdot \vec{A} = 0. \tag{21}$$

This means we have only two independent components of the potentials left. The equations of motion are given by the wave equation (14) with $\vec{j} = 0$,

$$\Box\vec{A} = 0. \tag{22}$$

We can now write any solution of these equations for $\vec{A}$ in terms of a spatial Fourier transform

$$\vec{A}(t, \vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \vec{a}(t, \vec{k}) \exp(\mathrm{i}\vec{k} \cdot \vec{x}). \tag{23}$$

The radiation-gauge constraint (22) translates to

$$\vec{k} \cdot \vec{a}(t, \vec{k}) = 0, \tag{24}$$

i.e., all plane-wave modes for the vector potential must be transverse. For a given $\vec{k} \neq 0$ we thus define two orthonormal arbitrary polarization vectors $\vec{\epsilon}_\lambda(\vec{k})$, $\lambda \in \{1, 2\}$, perpendicular to $\vec{k}$

$$\vec{\epsilon}_1(\vec{k}) \times \vec{\epsilon}_2(\vec{k}) = \frac{\vec{k}}{|\vec{k}|}. \tag{25}$$

So we can as well write (23) in the form

$$\vec{A}(t, \vec{x}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \vec{\epsilon}_\lambda(\vec{k}) a_\lambda(t, \vec{k}) \exp(\mathrm{i}\vec{k} \cdot \vec{x}). \tag{26}$$

Finally we have to fulfill also the wave equation (22), which leads to

$$\partial_t^2 a_\lambda(t,\vec{k}) + \vec{k}^2 a_\lambda(t,\vec{k}) = 0. \tag{27}$$

The general solution for these mode equations is given by

$$a_\lambda(t,\vec{k}) = A_\lambda^{(+)}(\vec{k})\exp(-\mathrm{i}\omega_{\vec{k}} t) + (-1)^\lambda A_\lambda^{(-)}(-\vec{k})\exp(+\mathrm{i}\omega_{\vec{k}} t), \quad \omega_{\vec{k}} = |\vec{k}|. \tag{28}$$

The conventional choice of $(-\vec{k})$ as the argument of $A_\lambda^{(-)}$ and the pre-factor $(-1)^\lambda$ will become clear in a moment. Then we have

$$\vec{A}(t,\vec{x}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \vec{\epsilon}_\lambda(\vec{k}) \exp(\mathrm{i}\vec{k}\cdot\vec{x}) \left[A_\lambda^{(+)}(\vec{k})\exp(-\mathrm{i}\omega_{\vec{k}} t) + (-1)^\lambda A_\lambda^{(-)}(-\vec{k})\exp(+\mathrm{i}\omega_{\vec{k}} t)\right]. \tag{29}$$

Now substituting $\vec{k} \to -\vec{k}$ leads to

$$\begin{split}\vec{A}(t,\vec{x}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \Big[&\vec{\epsilon}_\lambda(\vec{k}) A_\lambda^{(+)}(\vec{k}) \exp(-\mathrm{i}\omega_{\vec{k}} t)\exp(\mathrm{i}\vec{k}\cdot\vec{x}) \\ &+ (-1)^\lambda \vec{\epsilon}_\lambda(-\vec{k}) A_\lambda^{(-)}(+\vec{k}) \exp(-\mathrm{i}\vec{k}\cdot\vec{x})\exp(+\mathrm{i}\omega_{\vec{k}} t)\Big].\end{split} \tag{30}$$

If we now adapt the additional condition

$$\vec{\epsilon}_\lambda(-\vec{k}) = (-1)^\lambda \vec{\epsilon}_\lambda(+\vec{K}), \tag{31}$$

which is compatible with (25) we can write

$$\begin{split}\vec{A}(t,\vec{x}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \Big[&\vec{\epsilon}_\lambda(\vec{k}) A_\lambda^{(+)}(\vec{k}) \exp(-\mathrm{i}\omega_{\vec{k}} t)\exp(\mathrm{i}\vec{k}\cdot\vec{x}) \\ &+ \vec{\epsilon}_\lambda(\vec{k}) A_\lambda^{(-)}(+\vec{k}) \exp(-\mathrm{i}\vec{k}\cdot\vec{x})\exp(+\mathrm{i}\omega_{\vec{k}} t)\Big].\end{split} \tag{32}$$

Finally since the Maxwell fields and thus also the potential are real vector fields, we must have $A_\lambda^{(-)}(\vec{k}) = [A_\lambda^{(+)}(\vec{k})]^*$, and the final plane-wave representation of the general free vector potential reads

$$\vec{A}(t,\vec{x}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \vec{\epsilon}_\lambda(\vec{k}) \left[A_\lambda(\vec{k})\exp[-\mathrm{i}(\omega_{\vec{k}} t - \vec{k}\cdot x)] + A_\lambda^*(\vec{k})\exp[+\mathrm{i}(\omega_{\vec{k}} t - \vec{k}\cdot x)]\right]. \tag{33}$$

This can be interpreted as the superposition of continuously many harmonic oscillators labeled by the wave vectors $\vec{k} \in \mathbb{R}^3$ and polarization labels $\lambda \in \{1,2\}$.

The electric and magnetic field are now given by (5) and (3), respectively. With the radiation-gauge constraints (21) we find

$$\vec{E}(t,\vec{x}) = \sum_{\lambda=1} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \mathrm{i}\omega_{\vec{k}} \vec{\epsilon}_\lambda(\vec{k}) \left[A_\lambda(\vec{k})\exp[-\mathrm{i}(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x}) - A_\lambda^*(\vec{k})\exp[+\mathrm{i}(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})]\right], \tag{34}$$

$$\vec{B}(t,\vec{x}) = \sum_{\lambda=1} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \mathrm{i}\vec{k} \times \vec{\epsilon}_\lambda(\vec{k}) \left[A_\lambda(\vec{k})\exp[-\mathrm{i}(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x}) - A_\lambda^*(\vec{k})\exp[+\mathrm{i}(\omega_{\vec{k}} t - \vec{k}\cdot\vec{x})]\right]. \tag{35}$$

Now we see that this is just the superposition of an uncountable infinite set of harmonic oscillators labelled by λ and $\vec{k}$.

For the quantization we need a formulation in terms of the canonical formalism for fields, i.e., in terms of an **action functional**, from which the equations of motion occur via **Hamilton's principle of least action**. We start with the Lagrange formulation for our gauge-fixed vector potential. In this formulation we have to take the fields as dynamical quantities, i.e., we have to interpret the position arguments as labels for continuously many dynamical degrees of freedom. Thus the Lagrange function is given in terms of a Lagrange density that is a function of the fields and the first derivatives of the fields with respect to t and $\vec{x}$. Since the field equations are linear, they must be quadratic in the fields $\vec{A}$ and the spacetime derivatives. Since finally the Hamilton function, derived from the Lagrangian in the canonical formalism, should be gauge invariant, the action should be gauge invariant. Now we can use our above conventional treatment as a hint, how to build the Lagrangian, i.e., it should be a function of the gauge-independent fields $\vec{E} = -\dot{\vec{A}}$ and $\vec{B} = \vec{\nabla} \times \vec{A}$. Further our equation of motion is simply the homogeneous wave equation $\Box \vec{A} = 0$, and thus there are indeed only derivatives of the field in the Lagrangian. Last but not least it should be a scalar under rotations. This leaves the following form for the Lagrangian

$$\mathscr{L} = \frac{1}{2}\dot{\vec{A}}^2 + \alpha(\vec{\nabla} \times \vec{A})^2. \tag{36}$$

We have to determine the constant α such as to get the wave equation as the stationary point of the action functional

$$S[\vec{A}] = \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} \mathscr{L}. \tag{37}$$

The variation of the Lagrange density reads

$$\delta\mathscr{L} = \dot{\vec{A}} \cdot \partial_t \delta\vec{A} + 2\alpha(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\delta\vec{A}) = \dot{\vec{A}} \cdot \partial_t \delta\vec{A} + 2\alpha \left\{ \vec{\nabla} \cdot [\delta\vec{A} \times (\vec{\nabla} \times \vec{A})] + \delta\vec{A} \cdot [\vec{\nabla} \times (\vec{\nabla} \times \vec{A})] \right\}. \tag{38}$$

Plugging this into the integral for δS the total divergence vanishes due to Gauss's integral theorem, i.e., we get after integrating the first term by part with respect to t, using that in Hamilton's principle $\delta\vec{A} = 0$ at $t \in \{t_1, t_2\}$

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\ \delta\vec{A} \cdot \left[-\partial_t^2 \vec{A} + 2\alpha \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) \right]. \tag{39}$$

Now we still have to ensure the radiation-gauge constraint $\vec{\nabla} \cdot \vec{A}$ as an additional condition, but then $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = -\Delta\vec{A}$, and thus to make $\delta S = 0$ for fields fulfilling the homogeneous wave equation, we have to set $\alpha = -1/2$, because then

$$\delta S = \int_{t_1}^{t_2} \mathrm{d}t \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\ \delta\vec{A} \cdot \left(-\partial_t^2 \vec{A} + \Delta\vec{A} \right) \stackrel{!}{=} 0 \tag{40}$$

indeed implies $\Box \vec{A} = \partial_t^2 \vec{A} + \Delta\vec{A} = 0$.

Finally have

$$\mathscr{L} = \frac{1}{2}\dot{\vec{A}}^2 - \frac{1}{2}(\vec{\nabla} \times \vec{A})^2. \tag{41}$$

The Hamilton density is given by

$$\mathscr{H} = \vec{\Pi} \cdot \dot{\vec{A}} - \mathscr{L} \quad \text{with} \quad \vec{\Pi} = \frac{\partial \mathscr{L}}{\partial \dot{\vec{A}}} = \dot{\vec{A}}. \tag{42}$$

Finally this gives

$$\mathscr{H} = \frac{1}{2}\vec{\Pi}^2 + \frac{1}{2}(\vec{\nabla} \times \vec{A})^2. \tag{43}$$

4 Quantization of the free electromagnetic field

Now it is apparently easy to canonically quantize the free electromagnetic field, but there is one more subtlety. In the previous section we have written down the Lagrangian (41) under the radiation-gauge constraint that $\vec{\nabla} \cdot \vec{A} = 0$, and thus we cannot simply make $\vec{A}$ and $\vec{\Pi} = \dot{\vec{A}}$ operators and impose the canonical commutation relations, but we must do this only for the two independent transverse components. This transversality condition is, however, not so easy to impose directly on the fields. On the other hand we can directly work with the Fourier decomposition (33) and just quantize by making $A_\lambda(\vec{k})$ operators. For reasons, that will become clear in a moment, we also renormalize these Fourier components and thus make the substitution

$$A_\lambda(\vec{k}) \rightarrow \sqrt{\frac{(2\pi)^3}{2\omega_{\vec{k}}}} \mathbf{a}_\lambda(\vec{k}), \tag{44}$$

where upright bold-face symbols indicate operator-valued quantities. Thus we make the ansatz

$$\vec{\mathbf{A}}(t,\vec{x}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}} \vec{\epsilon}_\lambda(\vec{k}) \left[\mathbf{a}_\lambda(\vec{k}) \exp(-\mathrm{i}k \cdot x) + \mathbf{a}_\lambda^\dagger(\vec{k}) \exp(\mathrm{i}k \cdot x)\right]_{k_0 = \omega_{\vec{k}}}. \tag{45}$$

Here we have used the abbreviation $k \cdot x = k_0 t - \vec{k} \cdot \vec{x}$, borrowed from the relativistic four-vector formalism. Now we have a set of independent operators $\mathbf{a}_\lambda(\vec{k})$ and have to find the commutation relations for these field operators to be able to build the operator algebra.

We can assume to have the canonical commutation relations, *before* we have solved for the equations of motion with the transversality constraint from fixing the gauge, i.e., we simply assume

$$\left[\mathbf{A}_j(t,\vec{x}_1), \mathbf{A}_k(t,\vec{x}_2)\right] = \left[\mathbf{\Pi}_j(t,\vec{x}_1), \mathbf{\Pi}_k(t,\vec{x}_2)\right] = 0, \quad \left[\mathbf{A}_j(t,\vec{x}_1), \mathbf{\Pi}_k(t,\vec{x}_2)\right] = \mathrm{i}\delta_{jk}\delta^{(3)}(\vec{x}_1 - \vec{x}_2). \tag{46}$$

Then we can write down the Hamiltonian, setting operators for the classical fields into the Hamilton density (43)

$$\mathbf{H}' = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\, \frac{1}{2}\left[\vec{\mathbf{\Pi}}^2 + (\vec{\nabla} \times \vec{\mathbf{A}})^2\right]. \tag{47}$$

We have written a prime to the Hamilton operator, because we shall see below that this is not yet the final form, because we have a hidden operator-ordering problem, when we try to plug in the solution (45) into the integrand in (47). We shall, however, see that the necessary solution of this operator ordering problem does not invalidate the following formal evaluation of the commutators for the operator equations of motion of the field operators, which of course should be the same as the classical equations, because we deal with linear equations of motion.

Indeed, using the canonical commutation relations (46) and (47) one finds after a simple but lengthy calculation that the **Heisenberg equations of motion** for the field operators indeed come out right, i.e.,

$$\partial_t \vec{\mathbf{\Pi}}(t,\vec{x}) = \frac{1}{\mathrm{i}}\left[\vec{\mathbf{\Pi}}(t,\vec{x}),\mathbf{H}'\right] = -\vec{\nabla}\times[\vec{\nabla}\times\vec{\mathbf{A}}(t,\vec{x})], \quad \partial_t \vec{\mathbf{A}}(t,\vec{x}) = \left[\vec{\mathbf{A}}(t,\vec{x}),\mathbf{H}\right] = \vec{\mathbf{\Pi}}. \tag{48}$$

Now we have to solve this set of equations of motion under the radiation-gauge contraint

$$\vec{\nabla}\cdot\vec{\mathbf{A}}(t,\vec{x}) = 0. \tag{49}$$

and are led to (45). As we shall show now, this solution modifies the commutation relations (46) so that they become compatible with this constraint. To this end we introduce the relativistic mode functions

$$\vec{u}_{\lambda,\vec{k}}(t,\vec{x}) = \frac{1}{\sqrt{(2\pi)^3 2\omega_{\vec{k}}}}\vec{\epsilon}_\lambda(\vec{k})\exp(-\mathrm{i}\omega_{\vec{k}} t + \mathrm{i}\vec{k}\cdot\vec{x}), \quad \omega_{\vec{k}} = |\vec{k}|. \tag{50}$$

They fulfill the orthonormality relations

$$\begin{aligned} &\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,\vec{u}_{\lambda,\vec{k}}(t,\vec{x})\cdot\mathrm{i}\overleftrightarrow{\partial_t}\,\vec{u}_{\lambda',\vec{k}'}(t,\vec{x}) = 0, \\ &\int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,\vec{u}^*_{\lambda,\vec{k}}(t,\vec{x})\cdot\mathrm{i}\overleftrightarrow{\partial_t}\,\vec{u}_{\lambda',\vec{k}'}(t,\vec{x}) = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}'), \end{aligned} \tag{51}$$

where we define

$$A(t,\vec{x})\overleftrightarrow{\partial_t}B(t,\vec{x}) = A(t,\vec{x})\partial_t B(t,\vec{x}) - [\partial_t A(t,\vec{x})]B(t,\vec{x}). \tag{52}$$

With these orthonormality relations we can reverse the Fourier decomposition (45)

$$\mathbf{a}_\lambda(\vec{k}) = \mathrm{i}\int_{\mathbb{R}^3}\mathrm{d}^3\vec{x}\;\vec{u}^*_{\lambda,\vec{k}}(t,\vec{x})\mathrm{i}\overleftrightarrow{\partial_t}\,\vec{\mathbf{A}}(t,\vec{x}). \tag{53}$$

Using again the commutator relations (46) we find the commutators

$$\begin{aligned} &\left[\mathbf{a}_\lambda(\vec{k}),\mathbf{a}_{\lambda'}(\vec{k}')\right] = 0, \\ &\left[\mathbf{a}_\lambda(\vec{k}),\mathbf{a}^\dagger_{\lambda'}(\vec{k}')\right] = \delta_{\lambda\lambda'}\delta^{(3)}(\vec{k}-\vec{k}'). \end{aligned} \tag{54}$$

This means that the quantized free electromagnetic field is indeed equivalent to an infinite uncountable set of independent harmonic oscillators, for each *transverse* Fourier mode of the field labelled by λ and $\vec{k}$.

Now we have to derive the commutator relations for the field-operator solutions (45) and their consistency with (53), because to derive this latter equation, we have used the commutator relations (46), which are only valid before solving the equations of motion including the radiation-gauge constraint. Indeed, the only non-trivial commutator is

$$\left[\mathbf{A}_i(t,\vec{x}),\mathbf{\Pi}_j(t,\vec{x}')\right] = \left[\mathbf{A}_i(t,\vec{x}),\dot{\mathbf{A}}_j(t,\vec{x}')\right] = \mathrm{i}\delta^{(\perp)}_{ij}(\vec{x}-\vec{x}'), \tag{55}$$

where we have used the commutation relations (54), plugging the mode decomposition (45) into the arguments of the commutator and the relation

$$\sum_{\lambda=1}^{2}\vec{\epsilon}_{i\lambda}(\vec{k})\cdot\epsilon_{j\lambda}(\vec{k}) = \delta_{ij} - \frac{k_i k_j}{\vec{k}^2} = P^{(\perp)}_{ij}(\vec{k}). \tag{56}$$

The “transverse δ distribution” is then defined via the Fourier transformation

$$\delta_{ij}^{(\perp)}(\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \exp(\mathrm{i}\vec{k}\cdot\vec{x}) P_{ij}^{(\perp)}(\vec{k}). \tag{57}$$

Here, only the second part of the projector (56) to the transverse wave modes,

$$\delta_{ij}^{(\|)}(\vec{x}) = \int_{\mathbb{R}^3} \frac{\mathrm{d}^3\vec{k}}{(2\pi)^3} \exp(\mathrm{i}\vec{k}\cdot\vec{x}) \frac{k_i k_j}{\vec{k}^2}. \tag{58}$$

is non-trivial. To solve it we use

$$\Delta\delta_{ij}^{(\|)}(\vec{x}) = -\partial_i\partial_j\delta^{(3)}(\vec{x}), \tag{59}$$

which is easily derived from (58). From electrostatics we know, how to invert the Laplace operator:

$$\begin{aligned} \delta_{ij}^{(\|)}(\vec{x}) &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,' \frac{\partial_i'\partial_j'\delta^{(3)}(\vec{x}-\vec{x}\,')}{4\pi|\vec{x}-\vec{x}\,'|} \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,'\delta^{(3)}(\vec{x}-\vec{x}\,')\partial_i'\partial_j' \frac{q}{4\pi|\vec{x}-\vec{x}\,'|} \\ &= \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x}\,'\delta^{(3)}(\vec{x}-\vec{x}\,')\partial_i\partial_j \frac{q}{4\pi|\vec{x}-\vec{x}\,'|} \\ &= \partial_i\partial_j \frac{1}{4\pi|\vec{x}|} \;\Rightarrow\; \delta_{ij}^{(\perp)}(\vec{x}) = \delta_{ij}\delta^{(3)}(\vec{x}) - \partial_i\partial_j \frac{1}{4\pi|\vec{x}|}. \end{aligned} \tag{60}$$

Now it is also clear that the commutation relation (56) is consistent with both the commutation relations (54) and the Heisenberg equations of motion (48), because the additional term $\delta_{ij}^{(\|)}$ in (55) compared to the naive canonical commutators (46), because its contribution to the commutators cancel due to the transversality of the solution (45).

Now we can build the Hilbert space of the quantized free electromagnetic field in the usual way, because we know it already from the quantum mechanics of the harmonic oscillator. For each mode, defined by the polarization label λ and the wave number $\vec{k}$ the energy eigenmodes of the corresponding harmonic oscillator are the eigenvalues of the number operator

$$\mathbf{N}_\lambda(\vec{k}) = \mathbf{a}_\lambda^\dagger(\vec{k})\mathbf{a}_\lambda(\vec{k}). \tag{61}$$

The eigenvalues of each number operator are the set $\mathbb{N}_0 = \{0,1,2,\ldots\}$. What is counted here are **oscillator quanta**, and these define in the modern sense of the here discussed **quantum electrodynamics** what a **photon** is. A photon is given by its polarization λ and its wave number $\vec{k}$ and the corresponding state is the corresponding eigenstate of the number operator (61).

We also know from quantum mechanics that a complete orthonormal eigenbasis for each mode is given by the orthonormal vectors

$$|N_\lambda(\vec{k})\rangle = \frac{\left[\mathbf{a}_\lambda^\dagger(\vec{k})\right]^{N_\lambda(\vec{k})}}{\sqrt{N_\lambda(\vec{k})!}} |\Omega\rangle, \tag{62}$$

where $|\Omega\rangle$ is the ground state defined by

$$\mathbf{a}_\lambda(\vec{k})|\Omega\rangle = 0. \tag{63}$$

Because of the commutation relations (54) number operators (61) with different λ and $\vec{k}$ commute, and thus they can all be simultaneously diagonalized, i.e., have a common eigenbasis, the **Fock basis**

$$|\{N_{\lambda_i}(\vec{k}_j)\}_{ij}\rangle = \prod_{i,j} \frac{\left[\mathbf{a}^\dagger_{\lambda_i}(\vec{k}_j)\right]^{N_{\lambda_i}(\vec{k}_j)}}{\sqrt{N_{\lambda_i}(\vec{k}_j)!}} |\Omega\rangle, \tag{64}$$

where i and j run over any *countable* set and $|\Omega\rangle$, the "**vacuum state**" (representing the state that no photons are present) is defined such that (63) should hold for any $\lambda \in \{1,2\}$ and $\vec{k} \in \mathbb{R}^3$. We also know from the quantum theory of harmonic oscillators that $\mathbf{a}^\dagger_\lambda(\vec{k})$ creates a photon, i.e., applied to an Fock-basis state (64) it enhances the occupation number $N_\lambda(\vec{k})$ by one, while $\mathbf{a}_\lambda(\vec{k})$, lowers it by one and thus annihilates a photon. That's why these operators are called **creation and annihilation operators**. Due to the commutability of these operators it is clear that we cannot distinguish individual photons, i.e., photons with the same λ and $\vec{k}$ are **indistinguishable**, and the order of the operator product in the definition of (64) is irrelevant, because we used **commutation relations** to quantize the electromagnetic field. The Fock state (64) obviously does not change when interchanging any two creation operators in the product and thus photons are described as **bosons**.
There is still one more subtlety left to be discussed. The Hamilton operator (47) evaluates to

$$\mathbf{H}' = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\, \frac{\omega_{\vec{k}}}{2} \left[\mathbf{a}^\dagger_\lambda(\vec{k})\mathbf{a}_\lambda(\vec{k}) + \mathbf{a}_\lambda(\vec{k})\mathbf{a}^\dagger_\lambda(\vec{k})\right]. \tag{65}$$

Here we have the problem that the ordering of the operators is ill defined, because of the singular commutation relation for the annihilation and creation operators (54).
This can be cured somewhat by quantizing the electromagnetic field in a finite volume. Since we want to describe moving plane waves we must impose periodic boundary conditions[1]. Taking a cube of length L everything stays the same as above but the wave numbers are restricted to the discrete set

$$\vec{k} \in \frac{2\pi}{L}\mathbb{Z}^3. \tag{66}$$

The singular commutation relations (55) become regularized, i.e., the last commutator now reads

$$\left[\mathbf{a}_\lambda(\vec{k}), \mathbf{a}^\dagger_{\lambda'}(\vec{k}')\right] = \delta_{\lambda\lambda'}\delta_{\vec{k},\vec{k}'}, \tag{67}$$

with an innocent Kronecker δ instead of the Dirac-δ distribution in (54). Then we can rewrite the second term in the integrand in (65) as

$$\mathbf{a}_\lambda(\vec{k})\mathbf{a}^\dagger_\lambda(\vec{k}) = \left[\mathbf{a}_\lambda(\vec{k}), \mathbf{a}^\dagger_\lambda(\vec{k})\right] + \mathbf{a}^\dagger_\lambda(\vec{k})\mathbf{a}_\lambda(\vec{k}) = 1 + \mathbf{a}^\dagger_\lambda(\vec{k})\mathbf{a}_\lambda(\vec{k}). \tag{68}$$

Also the integral in (65) becomes a sum, the constant term in (68) still diverges since we sum over the infinitely many wave numbers (66). On the other hand, so far we have only needed commutator relations with $\mathbf{H}$ to show that the Heisenberg equations of motion are compatible with the corresponding classical equations of motion of the electromagnetic field, and thus we can simply omit these disturbing

[1] Indeed the very same formalism holds for quantization of the electromagnetic field in a cavity, where the photons are confined due to ideal reflecing walls, leading to standing-wave modes.

terms, i.e., we subtract an undefined diverging quantity proportional to the unit operator, which does not change the commutation relations of operators with the Hamiltonian. Thus, instead of (65) we can as well use the **normal ordered Hamiltonian**

$$\mathbf{H} =: \mathbf{H}' := \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\, \omega_{\vec{k}} \mathbf{a}_\lambda^\dagger \mathbf{a}_\lambda(\vec{k}) = \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\, \omega_{\vec{k}} \mathbf{N}_\lambda(\vec{k}). \tag{69}$$

This means that the occupation-number bosonic Fock states (62) are **energy eigenvectors** with

$$\mathbf{H} \left| \{N_{\lambda_i}(\vec{k}_j)\}_{i,j} \right\rangle = \left[\sum_{i,j} \omega_{\vec{k}_j} N_{\lambda_i}(\vec{k}_j) \right] \left| \{N_{\lambda_i}(\vec{k}_j)\}_{i,j} \right\rangle, \tag{70}$$

i.e., each photon of wave number $\vec{k}$ contributes an amount of $\omega_{\vec{k}} = |\vec{k}|$ of energy to the total energy. The colons in (69) denote **normal ordering**, i.e., in the operator product of creation and annihilation operators we write all creation operators to the very left and all annihilation operators to the very right. For any countable set of λ's and $\vec{k}$'s the order of the creation and annihilation operators among themselves is irrelevant, because of the Bose-commutator relations for these operators.

From electrodynamics we also know that the electromagnetic field carries not only energy but also momentum, and the momentum density is given by the **Poynting vector** $\vec{S} = \vec{E} \times \vec{B}$. In the spirit of the canonical-quantization heuristics, we simply set field operators but warned by the example of the Hamiltonian we fix the operator ordering in terms of the creation and annihilation operator by normal ordering, i.e., we define the operator of **total quantized field momentum** as

$$\vec{\mathbf{P}} = \int_{\mathbb{R}^3} \mathrm{d}^3\vec{x} : \vec{\mathbf{E}}(t,\vec{x}) \times \vec{\mathbf{B}}(t,\vec{x}) := \sum_{\lambda=1}^{2} \int_{\mathbb{R}^3} \mathrm{d}^3\vec{k}\, \vec{k}\, \mathbf{N}_\lambda(\vec{k}), \tag{71}$$

i.e., the Fock states (64) are also eigenstates of the total field energy, and each photon with wave number $\vec{k}$ contributes an amount of momentum given by $\vec{k}$.

It is tempting to give the photons a **particle interpretation**, but this is pretty misleading. Although in the sense of energy and momentum the photons seem to behave like particles, i.e., for each wave mode they contribute a certain amount of quantized energy and momentum when one considers the occupation-number states as (generalized) states of a many-body system consisting of photons. Without going into the details, one must however say that this is problematic, because photons cannot be localized in a very fundamental sense. One can show that it is not possible to define a position operator for a photon [Jor78], which obeys the usual Heisenberg-commutator relations for particles. This is because the photon is "massless" in the sense of the dispersion relation $\omega_{\vec{k}}^2 - \vec{k}^2 = 0$. If one would interpret $\omega_{\vec{k}}$ as energy and $\vec{k}$ as momentum of a (classical) particle in the naive sense of Einstein-deBroglie quantization, this particle would be thus massless. But the above mentioned "no-go theorems" prevent us from defining a position of a photon. As we shall demonstrate in the next section, it makes much more sense to define the classical limit of the quantized electromagnetic field simply in the sense of classical electromagnetic fields.

5 Thermal radiation (Planck radiation)

Historically the discovery of the quantum nature of radiation and only somewhat later of all kinds of matter is due to problems occuring in thermodynamics and statistical physics to describe **thermal**

radiation in **thermal equilibrium**. As we have stressed from the very beginning of this article, the radiation field can be interpreted as an infinite set of harmonic oscillators in both the classical Maxwell theory and in quantum electrodynamics. In the classical theory equilibrium of radiation is reached by the persistent emission and absorption of radiation from the walls of a cavity, and this radiation field can be described by a (countable) set of harmonic oscillators in thermal equilibrium. Now, according to classical theory, any oscillator provides a mean energy of $k_{\text{B}}T$ according to the equipartition theorem, where k_{B} is the Boltzmann constant and T the temperature. In the following we shall use natural units where besides $\hbar = c = 1$ also $k_{\text{B}} = 1$, i.e., we measure temperatures in units of energy (in atomic physics the typical energy scale is electron volts, eV, in high-energy physics MeV, GeV or even TeV). In any case since we have an infinite amount of harmonic oscillators the mean energy of the radiation field in thermal equilibrium diverges. This is the famous **UV catastrophe** of classical thermodynamics applied to the electromagnetic radiation field.

In our quantum-field theoretical framework, it is of course no problem to derive the correct energy spectrum, i.e., **Planck's radiation law**. To that purpose we use the box quantization, leading to a discrete set of possible wave numbers of the quantized radiation field. Since we are interested in the thermodynamic limit only, we can conveniently use the box with periodic boundary conditions, leading to the discretized wave numbers given by (66). The analysis for an ideally reflecting cavity, leading to "rigid boundary conditions" and standing wave modes ends up with the identical result in the thermodynamic limit.

The appriate statistical operator for the here addressed situation is the **canonical operator** for the radiation field,

$$\mathbf{R} = \frac{1}{Z}\exp(-\beta \mathbf{H}), \quad Z = \text{Tr}\,\exp(-\beta \mathbf{H}), \tag{72}$$

where β is the Lagrange parameter in the **maximum-entropy variational principle** contraining the mean radiation energy in the cavity to a certain value. As we shall see later, it is related to the temperature of the cavity walls by $\beta = 1/T$.

For our purposes we calulate a somewhat generalized partition sum, which we shall use as a generating functional for the mean energy per mode or the mean occupation number of photons in a given mode:

$$Z[\alpha] = \text{Tr}\exp\left(-\sum_{\lambda,\vec{k}} \alpha_\lambda(\vec{k})\mathbf{N}_\lambda(\vec{k})\right). \tag{73}$$

At the end of the calculation we also get the proper partition sum in (72) via

$$Z = Z[\alpha_{\text{eq}}] \quad \text{with} \quad \alpha_{\text{eq}\lambda}(\vec{k}) = \beta\omega_{\vec{k}}. \tag{74}$$

To evaluate the trace, we conveniently use the occupation-number Fock states (64). For our discrete set of wave numbers there is no quibble about how to calculate it:

$$Z[\alpha] = \prod_{\lambda,\vec{k}} \sum_{N_\lambda(\vec{k})=0}^{\infty} \exp\left(-\sum_{\lambda,\vec{k}} \alpha_\lambda(\vec{k})\mathbf{N}_\lambda(\vec{k})\right) = \prod_{\lambda,\vec{k}} \frac{1}{1-\exp[-\alpha_\lambda(\vec{k})]}. \tag{75}$$

Now it is more convenient to define the functional

$$\Omega[\alpha] = \ln Z[\alpha] = -\sum_{\lambda,\vec{k}} \ln\left\{1-\exp[-\alpha_\lambda(\vec{k})]\right\}. \tag{76}$$

Now the mean equibrium number of photons in the field mode defined by $(\vec{k}, \lambda)$ is after some simple calculation given by

$$\left\langle N_\lambda(\vec{k})\right\rangle = \mathrm{Tr}[\mathbf{R}\mathbf{N}_\lambda(\vec{k})] = \left[-\frac{\delta\Omega[\alpha]}{\delta\alpha_\lambda(\vec{k})}\right]_{\alpha=\alpha_{\mathrm{eq}}} = \frac{1}{\exp(\beta\omega_{\vec{k}})-1} = f_{\mathrm{B}}(\omega_{\vec{k}}, T) \tag{77}$$

and the mean radiation energy in this mode by

$$\left\langle \epsilon_\lambda(\vec{k})\right\rangle = \omega_{\vec{k}} f_{\mathrm{B}}(\omega_{\vec{k}}, T). \tag{78}$$

Now we can take the thermodynamic limit by making the box large such that the sum over the wave numbers in (76) can be approximated by an integral over a continuum of wave numbers. To this end we just have to count, how many states are in a small wave-number volume $\mathrm{d}^3\vec{k}$ these are

$$\mathrm{d}^3\vec{k}\rho(\vec{k}) = \mathrm{d}^3\vec{k}\frac{L^3}{(2\pi)^3} = \mathrm{d}^3\vec{k}\frac{V}{(2\pi)^3}. \tag{79}$$

So we can calculate the partition sum as

$$\Omega(\beta, V) = \ln Z = -\sum_{\vec{k},\lambda} \ln\left[1-\exp(-\beta\omega_{\vec{k}})\right] \simeq -\frac{2V}{(2\pi)^3}\int_{\mathbb{R}^3} \mathrm{d}^3\vec{k} \ln\left[1-\exp(-\beta\omega_{\vec{k}})\right] = \frac{\pi^2 V}{45\beta^3}. \tag{80}$$

The factor 2 in front of the integral is from the sum over the polarization states, labelled by $\lambda \in \{1,2\}$. Now we can evaluate all the "bulk properties" of the radiation with help of the **canonical potential** (80). First of all we note that the total mean energy, thermodynamically the **internal energy** of radiation is given by

$$U = \left\langle \sum_{\lambda,\vec{k}} \epsilon_\lambda(\vec{k})\right\rangle = -\partial_\beta\Omega(\beta, V) = \frac{\pi^2 V}{15\beta^4}. \tag{81}$$

Now the internal energy is not yet expressed as a function of its natural **thermodynamical variables**. According to the 1st Law of Thermodynamics we have

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V, \tag{82}$$

where T is the temperature, S the entropy, and P the pressure of the radiation. So the natural variables are the entropy and the volume. The entropy is given by the **Gibbs-Duhem relation**,

$$S = -\mathrm{Tr}(\mathbf{R}\ln\mathbf{R}) = \mathrm{Tr}[\mathbf{R}(\ln Z + \beta\mathbf{H})] = \Omega + \beta U. \tag{83}$$

From this we find

$$\mathrm{d}S = \mathrm{d}V\partial_V\Omega + \mathrm{d}\beta(\partial_\beta\Omega + U) + \beta\mathrm{d}U \stackrel{(81)}{=} \mathrm{d}V\partial_V\Omega + \beta\mathrm{d}U. \tag{84}$$

Solving for $\mathrm{d}U$ we get

$$\mathrm{d}U = \frac{1}{\beta}\mathrm{d}S - \frac{\partial_V\Omega}{\beta}\mathrm{d}V. \tag{85}$$

Comparing with the 1st Law (82) we find

$$T = \frac{1}{\beta}, \quad P = \frac{\partial_V \Omega}{\beta} \overset{(80)}{=} \frac{\pi^2}{45\beta^4} = \frac{U}{3}. \tag{86}$$

Another useful thermodynamic potential is the **free energy**

$$F(T,V) = -T\Omega(1/T,V) = U - TS. \tag{87}$$

From this we find, using (82)

$$\mathrm{d}F = \mathrm{d}U - \mathrm{d}(TS) = -S\mathrm{d}T - p\mathrm{d}V, \tag{88}$$

and thus

$$S = -\partial_T F(T,V), \quad P = -\partial_V F(T,V). \tag{89}$$

According to (80) The entropy of the radiation is thus given by

$$S = +\partial_T [T\Omega(1/T,V)] = \partial_T \frac{\pi^2 V T^4}{45} = \frac{4\pi^2 T^3 V}{45}. \tag{90}$$

6 The classical limit: Coherent states

TBD

7 Proof of the existence of photons I: beam-splitter experiments

8 Proof of the existence of photons II: quantum beats

TBD

9 Parametric downconversion

based on [HM85]
TBD

10 Bell tests with polarization-entangled

TBD

11 Outlook

TBD

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