

The electromagnetic field in GR

Hendrik van Hees

February 23, 2022

1 Introduction and conventions

In this manuscript we consider classical electrodynamics in a given general-relativistic background spacetime. This is motivated by the aim to understand the three standard effects on electromagnetic waves within classical electrodynamics rather than the pretty hand-waving “naive photon picture” used in many textbooks. These three effects are

- the deflection of light by the Sun, which was one of the first confirmations of GR by the famous solar-eclipse expedition by Eddington et al in 1919, confirming Einstein’s prediction of the deflection angle from his final version of the theory of Nov. 1915.
- the gravitational red shift of spectral lines. This has also been predicted by Einstein in his first papers, and the famous “Einstein tower” in Potsdam was an observatory dedicated explicitly to confirm the gravitational red-shift by spectroscopy of the light of the Sun. The accuracy for this has not been sufficient at the time. The effect of gravitational red (or in this case rather blue shift) has then been successfully observed for the first time using γ -rays and the Mössbauer effect in the gravitational field of the Earth.
- the redshift-distance relation of far-distant objects (Hubble-Lemaitre Law), discovered famously by Hubble and has been predicted by Lemaitre based on the cosmological standard model described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric as a description of the large-scale structure of spacetime.

As it turns out, all that’s needed is the formulation of the electrodynamics in a static spacetime. As we shall see, also the case of the FLRW metric, which is not a static solution but describes “Hubble expansion”, can be deduced from this simpler static case.

We follow the sign conventions of [ABS75, Fli12]: The signature of the metric is $(+, -, -, -) = (1, 3)$ (west-coast convention), i.e., in flat Minkowski spacetime in Lorentzian coordinates, the metric components are given by $(\eta_{\mu\nu}) = \text{diag}(1, -1, -1, -1)$.

The Christoffel symbols are then given by

$$\Gamma_{\mu\nu}^{\rho} = \Gamma_{\nu\mu}^{\rho} = \frac{1}{2}(g^{\rho\sigma} \partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\rho} g_{\mu\nu}), \quad (1)$$

where the usual Einstein summation convention applies (over a pair of two equal indices, one upper and one lower, one has to sum from 0 to 3), defining the covariant derivatives of vector fields via

$$\nabla_{\mu} V^{\nu} = \partial_{\mu} V^{\nu} + \Gamma_{\mu\rho}^{\nu} V^{\rho}, \quad \nabla_{\mu} V_{\nu} = \partial_{\mu} V_{\nu} - \Gamma_{\mu\nu}^{\rho} V_{\rho}. \quad (2)$$

The Riemann curvature tensor is given via the Christoffel symbols by

$$R^\alpha{}_{\mu\beta\gamma} = \partial_\gamma \Gamma_{\mu\beta}^\alpha - \partial_\beta \Gamma_{\mu\gamma}^\alpha + \Gamma_{\gamma\nu}^\alpha \Gamma_{\mu\beta}^\nu - \Gamma_{\beta\nu}^\alpha \Gamma_{\mu\gamma}^\nu. \quad (3)$$

Then for any vector field's covariant components one has

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) V_\rho = -R_{\rho\sigma\mu\nu} V^\sigma. \quad (4)$$

The Ricci tensor is then defined by contraction of the 1st with the 3rd index, **Ricci-Tensor**

$$R_{\mu\nu} = R_{\nu\mu} = R^\alpha{}_{\mu\alpha\nu}. \quad (5)$$

One more contraction leads to the Ricci scalar,

$$\mathcal{R} = R^\mu{}_\mu. \quad (6)$$

With these sign conventions the Einstein field equations of the gravitational interaction reads

$$G_{\mu\nu} := R_{\mu\nu} - \frac{\mathcal{R}}{2} g_{\mu\nu} = -\chi T_{\mu\nu}, \quad (7)$$

where $T_{\mu\nu} = T_{\nu\mu}$ is the energy-momentum tensor of matter and radiation. Further $\chi = 8\pi G/c^4$.

Taking the trace of (7) shows that the Einstein equations can also be written as

$$R_{\mu\nu} = -\chi \left(T_{\mu\nu} - \frac{1}{2} T^\rho{}_\rho g_{\mu\nu} \right). \quad (8)$$

Finally the Bianchi identity,

$$\nabla^\mu G_{\mu\nu} = 0, \quad (9)$$

implies that the covariant divergence of the energy-momentum tensor has to vanish:

$$\nabla^\mu T_{\mu\nu} = 0. \quad (10)$$

Finally we define the Levi-Civita symbol such that

$$\epsilon^{0123} = +1, \quad \epsilon_{0123} = -1. \quad (11)$$

Othewise both $\epsilon^{\mu\nu\rho\sigma}$ and $\epsilon_{\mu\nu\rho\sigma}$ are completely anti-symmetric under exchange of any two indices. The components of the Levi-Civida “pseudo-tensor” are then defined by

$$\Delta^{\mu\nu\rho\sigma} = \frac{1}{\sqrt{-g}} \epsilon^{\mu\nu\rho\sigma}, \quad \Delta_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \quad \text{with} \quad g = \det(g_{\mu\nu}), \quad (12)$$

where $g = \det(g_{\mu\nu}) < 0$.

2 Electrodynamics in General Relativity

To generalize the classical electrodynamics from Minkowski space to the general relativistic spacetime model, we make use of the heuristic rule to write covariant derivatives ∇_μ acting on tensor components wrt. arbitrary coordinate bases for the partial derivatives ∂_μ in Minkowski spacetime in Lorentzian coordinates. This is possible without ambiguities as long as we deal with equations, where only first derivatives occur, because then there is no uncertainty concerning the order of the covariant derivatives, which is not determined from the expressions in Minkowski spacetime, because the partial derivatives commute. Thus in electrodynamics we just use the Maxwell equations for the field-strength tensor, which are of first order. The inhomogeneous Maxwell equations read, using Heaviside-Lorentz units for the electromagnetic quantities

$$\nabla_\mu F^{\mu\nu} = \frac{1}{c} j^\nu, \quad (13)$$

Here $F^{\mu\nu}$ are the contravariant components of the antisymmetric Faraday tensor, and j^ν the electric four-current density. Because of the antisymmetry $F^{\mu\nu} = -F^{\nu\mu}$ the covariant divergence in (13) can be written as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = \frac{1}{c} j^\nu. \quad (14)$$

The homogeneous Maxwell equations read

$$\nabla_\mu F_{\rho\sigma} + \nabla_\rho F_{\sigma\mu} + \nabla_\sigma F_{\mu\rho} \equiv \partial_\mu F_{\rho\sigma} + \partial_\rho F_{\sigma\mu} + \partial_\sigma F_{\mu\rho} = 0. \quad (15)$$

This can be expressed with the dual field-strength tensor

$$\tilde{F}_{\mu\nu} = \Delta_{\mu\nu\rho\sigma} F^{\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \quad (16)$$

as

$$\nabla^\mu \tilde{F}_{\mu\nu} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma} \nabla^\mu F^{\rho\sigma} = 0. \quad (17)$$

Here, the first equality follows because the covariant derivatives of the metric components vanish and the second, because the contraction with the Levi-Civita symbol reproduces the left-hand side of Eq. (15).

Because the covariant derivatives in (15) can be written with the partial derivatives, as in electrodynamics in Minkowski space these homogeneous Maxwell equations imply the existence of a **vector potential**, i.e., that the Faraday tensor can be written as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu, \quad (18)$$

where the second form follows from

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\rho A_\rho \quad (19)$$

via the symmetry of the Christoffel symbols under commutation of its lower indices, $\Gamma_{\mu\nu}^\rho = \Gamma_{\nu\mu}^\rho$, and

$$\nabla_\mu A_\nu = \partial_\mu A_\nu - \Gamma_{\mu\nu}^\rho A_\rho. \quad (20)$$

Further it is clear that the four-potential is defined only up to a gradient of a scalar field, i.e., the **gauge-transformed four-potential**,

$$A'_\mu = A_\mu + \partial_\mu \chi = A_\mu + \nabla_\mu \chi, \quad (21)$$

leads to the same Faraday tensor (18) as A_μ .

As in Minkowski space fixing the gauge (partially) by imposing the **Lorenz-gauge constraint**,

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} A^\mu) \stackrel{!}{=} 0 \quad (22)$$

simplifies the field equations for A_μ . Since (15) is satisfied identically by using (18), the remaining independent field equations are the inhomogeneous Maxwell equations (13). Using (18) in (13) leads to

$$g^{\alpha\beta} \nabla_\alpha F_{\beta\nu} = g^{\alpha\beta} \nabla_\alpha (\nabla_\beta A_\nu - \nabla_\nu A_\beta) = j_\nu. \quad (23)$$

Cf. (4) we have

$$(\nabla_\alpha \nabla_\nu - \nabla_\nu \nabla_\alpha) A_\beta = -R_{\beta\rho\alpha\nu} A^\rho, \quad (24)$$

and contraction with $g^{\alpha\beta}$ yields

$$j_\nu = g^{\alpha\beta} \nabla_\alpha (\nabla_\beta A_\nu - \nabla_\nu A_\beta) \quad (25)$$

$$= \square A_\nu - g^{\alpha\beta} (\nabla_\alpha \nabla_\nu - \nabla_\nu \nabla_\alpha + \nabla_\nu \nabla_\alpha) A_\beta \quad (26)$$

$$\stackrel{(4,5)}{=} \square A_\nu - \nabla_\nu \nabla_\alpha A^\alpha + R_{\nu\gamma} A^\gamma \quad (27)$$

$$\stackrel{(22)}{=} \square A_\nu + R_{\nu\gamma} A^\gamma. \quad (28)$$

Here we have used the Lorenz-gauge condition (22) only in the very last step. In general gauge (27) holds. As we see, we'd not been able to derive this equation from the equation in Minkowski spacetime, because the covariant derivatives do not commute, and a simple substitution of the partial derivatives by covariant derivatives is ambiguous. Indeed, starting with the equation in Minkowski space in the form

$$\square A_\nu - \partial_\nu \partial_\alpha A^\alpha = j_\nu \quad (\text{valid only in Minkowski space in pseudo-Cartesian coordinates!}) \quad (29)$$

and just making $\partial_\mu \rightarrow \nabla_\mu$, we'd have missed the additional term in (27) with the Ricci tensor. As the above derivation shows, without this term electromagnetic gauge invariance would get lost and the term related to general-relativistic space-time curvature must necessarily occur in the Maxwell equations written as 2nd-order differential equations for the four-potential.

3 Conformal invariance of the free Maxwell equations

The free Maxwell equations are invariant under conformal transformations, i.e., given an arbitrary function $\lambda = \lambda(q^\mu)$, setting

$$g_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}, \quad A_\mu = \tilde{A}_\mu, \quad (30)$$

it follows from (18)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu = \tilde{F}_{\mu\nu}. \quad (31)$$

Further we have

$$g^{\mu\nu} = \frac{1}{\lambda} \tilde{g}^{\mu\nu}, \quad (32)$$

because $(g^{\mu\nu})$ is the inverse matrix of $(g_{\mu\nu})$. Thus we get

$$F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma} = \frac{1}{\lambda^2} \tilde{g}^{\nu\sigma} \tilde{F}_{\rho\sigma} = \frac{1}{\lambda^2} \tilde{F}^{\mu\nu}, \quad (33)$$

where the usual tensor operations like lowering and raising indices for the transformed tensor components have to be always done with $\tilde{g}_{\mu\nu}$ or $\tilde{g}^{\mu\nu}$, respectively.

Evidently the homogeneous Maxwell equations (15) hold for $\tilde{F}_{\mu\nu}$ if and only if they hold for $F_{\mu\nu}$ since these equations are entirely independent of the metric.

For $j_\nu = 0$ also for the free Maxwell equations (14) we have

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = \frac{1}{\lambda^2 \sqrt{-\tilde{g}}} \partial_\mu (\sqrt{-\tilde{g}} \tilde{F}^{\mu\nu}) = 0. \quad (34)$$

Multiplication by λ yields

$$0 = \nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-\tilde{g}}} \partial_\mu (\sqrt{-\tilde{g}} \tilde{F}^{\mu\nu}) = \tilde{\nabla}_\mu \tilde{F}^{\mu\nu}, \quad (35)$$

where $\tilde{\nabla}_\mu$ is the covariant derivative, which is defined by the Christoffel symbols via $\tilde{g}_{\mu\nu}$.

For electrodynamics in the Friedmann-Lemaître-Robertson-Walker spacetimes of cosmology this implies a tremendous simplification, because by a specific choice of coordinates $(q^\mu) = (\tau, \chi, \vartheta, \varphi)$ the metric can be written as

$$(g_{\mu\nu}) = a^2(\tau) \text{diag}(1, -1, -S_K^2(\chi), -S_K^2(\chi) \sin^2 \vartheta), \quad (36)$$

i.e. it is given by the conformal transformation of the static metric

$$\tilde{g}_{\mu\nu} = \text{diag}(1, -1, -S_K^2(\chi), -S_K^2(\chi) \sin^2 \vartheta). \quad (37)$$

The functions S_K are defined by

$$S_K(\chi) = \begin{cases} \sin \chi & \text{für } K = 1, \\ \chi & \text{für } K = 0, \\ \sinh \chi & \text{für } K = -1. \end{cases} \quad (38)$$

Thus free electromagnetic fields can be written in terms of $F_{\mu\nu} = \tilde{F}_{\mu\nu}$, which is independent of the specific form of the scale parameter $a(\tau)$. Particularly for $K = 0$ (37) is the **Minkowski metric**, written in spatial spherical coordinates, $(\chi, \vartheta, \varphi)$. All solutions of the free Maxwell equations in Minkowski space are thus also solutions in all flat FLRW spacetimes. Particularly the plane waves $A_\mu \propto \exp(-ik_\mu x^\mu)$ with $k_\mu k^\mu = 0$ are solutions. In the next Sect. we show that this also holds approximately for the non-flat FLRW metrics, i.e., for $K \in \{-1, +1\}$.

4 The eikonal approximation

Exact solutions of the free Maxwell equations for $K \neq 0$ are not as easy to find as in Minkowski space since the equations for A_μ do not separate into wave equations for the components. One solution is the generalization of the multipole expansion to a large class of spacetimes defined by exact solutions of the Einstein field equations (such as Schwarzschild, Kerr, FLRW spacetimes) can be found in [CK74]. To derive the cosmological Hubble-Lemaître redshift of spectral lines of the electromagnetic waves emitted by stars, supernovae, galaxies, etc, it is however sufficient to use the **eikonal approximation**.

This corresponds to the approximation of wave optics by ray optics and is valid far away from the source and from all obstacles, i.e., for $L \gg \lambda$, where λ is the typical wave length of the considered electromagnetic wave and L the typical scales across which quantities like the index of refraction considerably change. For a detailed treatment of the eikonal approximation in flat Minkowski spacetime, see e.g., [Som54, LL96]. In GR another typical length scale is also the curvature scalar, R , i.e., we also have to assume that $\lambda \ll R$. Then we introduce the small parameter

$$\epsilon = \frac{\lambda}{\min(L, R)} \ll 1 \quad (39)$$

and make the ansatz for the four-potential,

$$A^\mu = \text{Re} \left\{ [a_0^\mu + a_1^\mu \epsilon + \mathcal{O}(\epsilon^2)] \exp\left(-i \frac{\psi}{\epsilon}\right) \right\}. \quad (40)$$

The parametric dependence of the phase on ϵ results from $k = 2\pi/\lambda$. For the following calculation we can first calculate with the complex exponential without taking its real part, which is allowed as long as we only deal with linear expressions in the A_μ and its derivatives with respect to the real coordinates, q^μ .

First we impose the Lorenz gauge condition (22). With our ansatz (40) we find

$$\nabla_\mu A^\mu = \frac{1}{\sqrt{-g}} \left\{ \partial_\mu [\sqrt{-g}(a_0^\mu + \epsilon a_1^\mu)] - \frac{i}{\epsilon} (a_0^\mu + \epsilon a_1^\mu) \partial_\mu \psi \right\} \exp\left(-i \frac{\psi}{\epsilon}\right) = 0. \quad (41)$$

In leading order $\mathcal{O}(1/\epsilon)$ we find

$$a_0^\mu \partial_\mu \psi = 0. \quad (42)$$

Expanding the phase ψ in a neighborhood of an arbitrary spacetime point to linear order in ϵ we find that

$$k_\mu = \partial_\mu \psi. \quad (43)$$

The wave vector changes via the curvature of spacetime in a similar way as it changes for an electromagnetic wave in a dispersive medium with a spacetime dependent **index of refraction**. This is the case particularly for electrodynamics in spacetimes that are conformal to a static spacetime as the FLRW spacetimes, for which $g_{\mu\nu} = \lambda \tilde{g}_{\mu\nu}$, where the $\tilde{g}_{\mu\nu}$ do not depend on the temporal coordinate and which can be brought in a form such that $g_{0j} = g_{j0} = 0$ for $j \in \{1, 2, 3\}$ by a choice of the coordinates (q^μ) [LL96].

Plugging now our ansatz (40) into the wave equation (28), we find that the leading order $\mathcal{O}(1/\epsilon^2)$ originates from taking the 2nd derivatives of the exponential factor contained in the generalized d'Alembert operator \square . This leads to

$$k_\mu k^\mu = g^{\mu\nu} (\partial_\mu \psi) (\partial_\nu \psi) = 0. \quad (44)$$

As expected, the wave vector turns out to be lightlike. Of course we can cancel the scale factor \tilde{a} in this equation, as expected from the conformal invariance of the free Maxwell equations, i.e., (44) is equivalent to

$$\tilde{g}^{\mu\nu} (\partial_\mu \psi) (\partial_\nu \psi) = 0. \quad (45)$$

Taking the covariant derivative of (44) we find

$$0 = \nabla_\mu (k_\nu k^\nu) = 2k^\nu \nabla_\mu k_\nu = 2k^\nu \nabla_\mu \nabla_\nu \psi = 2k^\nu \nabla_\nu \nabla_\mu \psi = 2k^\nu \nabla_\nu k_\mu. \quad (46)$$

Now the normal vectors of the hypersurfaces of constant phase, $\psi = \text{const}$, define the **Light rays** with the tangent vector k^μ . Parametrizing the light ray as $q^\mu(\lambda)$, where λ is an arbitrary world-line parameter, we have, choosing an appropriate normalization of λ ,

$$\dot{q}^\mu = k^\mu. \quad (47)$$

Here the dot means a derivative with respect to λ . From (46) it follows that the light ray is a **null geodesic**, because

$$\frac{D^2 q^\mu}{D\lambda^2} = \frac{Dk^\mu}{D\lambda} = \dot{q}^\nu \nabla_\nu k^\mu \stackrel{(47)}{=} k^\nu \nabla_\nu k^\mu \stackrel{(46)}{=} 0. \quad (48)$$

This means that the often used “naive photon picture” for the light propagation in curved spacetime is indeed correct in the sense of the eikonal approximation of classical wave optics, i.e., the light rays can be interpreted as trajectories of massless particles.

The geodesic equation can be derived from the action principle with the action

$$S[q] = \int d\lambda L = \frac{1}{2} \int d\lambda \tilde{g}_{\mu\nu} \dot{q}^\mu \dot{q}^\nu. \quad (49)$$

Since the Lagrangian L does not explicitly depend on the world-line parameter λ the “Hamiltonian”,

$$H = \dot{q}^\mu p_\mu - L = L = \text{const} \quad \text{with} \quad p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}, \quad (50)$$

is conserved for the null geodesic, obeying $\dot{q}_\mu \dot{q}^\mu = 0$, which thus can be imposed as the appropriate constraint for the motion of a massless particle, and λ is an affine parameter for this null geodesic.

Since further $\tilde{g}_{\mu\nu}$ is a static metric, i.e., τ is a cyclic coordinate, the corresponding canonical momentum is conserved along the light ray:

$$p_0 = \frac{\partial L}{\partial \dot{q}^0} = k_0 =: \tilde{\omega} = \text{const}. \quad (51)$$

We are now interested in the radial light rays, originating from a far-distant source. So we set $\psi = \psi(\tau, \chi)$. With the metric (37) the eikonal equation (45) then simplifies to

$$(\partial_\tau \psi)^2 - (\partial_\chi \psi)^2 = 0. \quad (52)$$

Because of (51) and because $k_0 = k^0 = \omega$, we have

$$\partial_\chi \psi = -\partial_\tau \psi = -\tilde{\omega}, \quad (53)$$

where we have chosen the negative square root, because we consider the case that the observer is at χ_o and the light source at $\chi = 0$. The light source itself we consider as radially symmetric with a surface at χ_s such that $\chi_2 \ll \chi$, so that the eikonal approximation is fulfilled at the position χ_o of the observer.

Now (53) is solved by

$$\psi(\tau, \chi) = \tilde{\omega}(\tau - \chi), \quad (54)$$

which indeed describes a light wave moving outwards from $\chi = 0$ to $\chi = \chi_o$. However, usually we describe the observers as **co-moving observers**. For these the FLRW metric is given in the form

$$(g_{\mu\nu}) = \text{diag}(1, -a^2(t), -a^2(t)S_K^2(\chi), -a^2(t)S_k^2 \sin^2 \vartheta), \quad (55)$$

where

$$d\tau = \frac{dt}{a(t)}, \quad a(t) = \tilde{a}[\tau(t)]. \quad (56)$$

Thus the frequency with respect to the co-moving coordinates is

$$\omega = \partial_t \psi = \frac{d\tau}{dt} \partial_\tau \psi = \frac{d\tau}{dt} \tilde{\omega} \stackrel{(56)}{=} \frac{\tilde{\omega}}{a(t)}. \quad (57)$$

Thus along the radial light beam we have

$$\omega a = \tilde{\omega} = \text{const.} \quad (58)$$

A light signal that is emitted at time t_e at $\chi = 0$ arrives the observer at the later time t_o at the obser at $\chi = 0$. With (58) we have for the corresponding frequencies at emission and observation

$$\omega_e a(t_e) = \omega_o a(t_o) \Rightarrow \frac{\omega_e}{\omega_o} = \frac{a(t_o)}{a(t_e)} = 1 + z. \quad (59)$$

Since the universe expands, i.e., $a(t_o) > a(t_e)$ we have $\omega_e > \omega_o$, i.e., the spectral lines are red-shifted compared to their properties at emission. Thus, interpreted within these comoving coordinates, i.e., for observers comoving with the “cosmic substrate”, the Hubble-Lemaître redshift is not a Doppler effect but is solely due to the time-dependence of the scale paratmeter $a(t)$.

For an observer on Earth this is not true, because as the measurement of the dipole component of the dependence of the temperature of the cosmic microwave background on the direction of observation shows, we are moving with about 370 km/s wrt. to the rest frame of the cosmic microwave background which defines the standard coordinates for co-moving observers for the FLRW spacetime (55).

To find the time t_o we have to solve the equation for the radial null geodesics, for which we can use the first integral (50)

$$ds^2 = dt^2 - a^2 d\chi^2 = 0, \quad (60)$$

i.e.,

$$\chi_o - \chi_s \simeq \chi_o = \int_{t_e}^{t_o} \frac{dt}{a(t)} = (\tau_o - \tau_e). \quad (61)$$

For a given $a(t)$, which is derived from the Friedmann equations for a given “energy content” of the universe, we can derive t_o for a given t_e .

The frequency shift for the more general case, where both the observer and the source are moving wrt. the co-moving frames of reference (i.e., the local rest frames of the cosmic micro-wave background radition) is given by writing the frequencies in a manifestly covariant form as $\omega_e = u_e^\mu k_\mu(t_e, \chi_s)$ and $\omega_o = u_o^\mu k_\mu(t_o, \chi_o)$, where

$$k_0 = k_t = \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial \tau} \frac{d\tau}{dt} = \frac{\tilde{\omega}}{a(t)}, \quad k_1 = k_\chi = \partial_\chi \psi = \tilde{\omega}, \quad k_2 = k_3 = 0. \quad (62)$$

Thus we have in manifestly covariant form for a moving source and observer

$$\frac{\omega_e}{\omega_o} = \frac{k_{e\mu} u_e^\mu}{k_{o\mu} u_o^\mu}. \quad (63)$$

For a co-moving observer and co-moving source we have $(u_e^\mu) = (u_o^\mu) = (1, 0, 0, 0)$ and (62) leads again to (59). For the general case of an arbitrarily moving source and observer (63) describes both the Hubble-Lemaître redshift and the Doppler effect due to the motion of source and observer wrt. the local rest frames of the cosmic microwave background radiation.

Only for not too far distant light sources one can approximately reinterpret the Hubble-Lemaître redshift as a kind of Doppler effect. To see this we expand the denominator in (59) as

$$1 + z = \frac{\omega_e}{\omega_o} = \frac{a(t_o)}{a(t_e)} \simeq \frac{a(t_o)}{a(t_o) - \Delta t \dot{a}(t_o)} \simeq 1 + \Delta t \dot{a}(t_o)/a(t_o) = 1 + H_o \Delta t, \quad (64)$$

where $H_o = \dot{a}(t_o)/a(t_o)$ is the **Hubble constant** and $\Delta t = t_o - t_e$. For this expansion to be valid we must have $z \simeq H_o \Delta t \ll 1$. On the other hand the distance between the light source and the observer is, according to the FLRW metric

$$r(t_o) = a(t_o) \chi_o \Rightarrow v_o = \dot{a}(t_o) \chi_o = H_o a(t_o) \chi_o. \quad (65)$$

Due to (61) we have

$$\chi_o = \int_{t_e}^{t_o} \frac{dt}{a(t)} \simeq \frac{t_o - t_e}{a(t_o)} = \frac{\Delta t}{a(t_o)} \quad (66)$$

and thus because of (65)

$$v_o \simeq H_o \Delta t. \quad (67)$$

In the same order of approximation this is indeed also the Doppler effect in Minkowski spacetime for an observer moving radially away from the light source with velocity v_o :

$$1 + z = \frac{\omega_e}{\omega_o} = \sqrt{\frac{1 + v_o}{1 - v_o}} \simeq 1 + v_o, \quad (68)$$

and this is in accordance with (64) due to (67), but this interpretation is valid only for $z \ll 1$.

5 Luminosity distance

Another important application of the above theory of the free electromagnetic field is the definition of the **luminosity distance**, which relates the measured energy-flux density of radiation from light sources of known luminosity (“standard candles”) with the distance of the source from the observer. In the context of cosmology important standard candles are the type Ia supernovae. In the following we shall derive the relation between the observed brightness and the distance parameter χ of the FLRW metric.

To that end we need the **energy-momentum tensor** of the free electromagnetic field. In GR it is found from the action functional of the free electromagnetic field,

$$S[A_\mu] = -\frac{1}{4} \int d^4 q \sqrt{-g} F_{\mu\nu} F^{\mu\nu} \quad (69)$$

by variation with respect to the metric $g_{\mu\nu}$ at fixed A_μ . According to (31) under this variation $\delta F_{\mu\nu} = 0$. Then we need the variation of $g = \det(g_{\mu\nu})$. Since the matrix $(g^{\mu\nu})$ is the inverse of $(g_{\mu\nu})$ we have

$$\delta g = \frac{\partial g}{\partial g_{\mu\nu}} \delta g_{\mu\nu} = g \delta g_{\mu\nu} g^{\mu\nu} \quad (70)$$

and thus

$$\delta \sqrt{-g} = -\frac{1}{2\sqrt{-g}} g g^{\mu\nu} \delta g_{\mu\nu} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}. \quad (71)$$

For the variation of $F^{\mu\nu}$ we also need the variation of $g^{\mu\nu}$. Because of $g_{\mu\nu} g^{\nu\rho} = \delta_{\mu}^{\rho}$ we have

$$g_{\mu\nu} \delta g^{\nu\rho} = -\delta g_{\mu\nu} g^{\nu\rho}. \quad (72)$$

Using the antisymmetry of $F_{\mu\nu}$ with this we get

$$\delta F^{\alpha\beta} = \delta(g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu}) = (\delta g^{\mu\alpha} g^{\nu\beta} + g^{\mu\alpha} \delta g^{\nu\beta}) F_{\mu\nu} = 2\delta g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} = -2F^{\mu\beta} g^{\nu\alpha} \delta g_{\mu\nu}. \quad (73)$$

The energy-momentum tensor $T^{\mu\nu}$ of any fields with an action S is given by

$$\delta S = -\frac{1}{2} \int d^4q \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu}. \quad (74)$$

Using (71) and (72) in the variation of the action for the electromagnetic field (69), after some calculation one obtains

$$T^{\mu\nu} = F^{\mu\alpha} F_{\alpha}^{\nu} + \frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} g^{\mu\nu}. \quad (75)$$

In the leading order of the eikonal approximation (44) in conformal coordinates the solution of plane waves propagating in radial direction reads

$$A_{\mu} = a_{0\mu} \cos[\tilde{\omega}(\tau - \chi)], \quad (76)$$

Since $k_{\mu} a_0^{\mu} = 0$ we choose the orientation of the spatial reference frame such that

$$(a_{0\mu}) = (0, 0, a_0, 0). \quad (77)$$

In this order of the eikonal approximation we can assume that $a_0 = \text{const.}$ After some calculations (cf. the Mathematica notebook in Appendix A)

$$\tilde{T}^{\mu\nu} = \tilde{\epsilon}(\tau, \chi) \tilde{u}^{\mu} \tilde{u}^{\nu} \quad \text{with} \quad \tilde{u}_{\mu} = (\tilde{k}_{\mu}/\tilde{\omega}) = (1, -1, 0, 0) \quad (78)$$

with the scalar field

$$\tilde{\epsilon}(\tau, \chi) = \frac{a_0^2 \tilde{\omega}^2 \sin^2[\tilde{\omega}(\tau - \chi)]}{\tilde{a}^2(\tau) S_K^2(\chi)}. \quad (79)$$

As can be easily checked, this approximation fulfills the local energy-conservation law,

$$\nabla_{\mu} \tilde{T}^{\mu\nu} = 0. \quad (80)$$

Further, because of $\tilde{T}^{\mu\nu} \propto k^{\mu} k^{\nu}$, the trace $g_{\mu\nu} \tilde{T}^{\mu\nu} = 0$ as also follows from Noether's theorem applied to the conformal invariance of the free electromagnetic field.

As usual due to the rapid oscillations of the \sin^2 factor only the averaged intensity $\langle T^{00} \rangle$ is observable. Since over one period of the wave $\tilde{a}(\tau)$ can be considered as constant, we only need to average over the \sin^2 factor,

$$\langle \tilde{\epsilon}(\tau, \chi) \rangle = \frac{a_0^2 \tilde{\omega}^2}{\tilde{a}^2(\tau) S_K^2(\chi)} \frac{1}{\tau_0} \int_0^{\tau_0} \sin^2[\tilde{\omega}(\tau + \chi)] = \frac{a_0^2 \tilde{\omega}^2}{2\tilde{a}^2(\tau) S_K^2(\chi)}. \quad (81)$$

Since it is a scalar field, we have $\tilde{\epsilon}(\tau, \chi) = \epsilon(t, \chi)$, and thus for a co-moving observer

$$\langle \epsilon(t, \chi) \rangle = \frac{a_0^2 \tilde{\omega}^2}{2a^2(t) S_K^2(\chi)} \quad (82)$$

and

$$(u^\mu) = \left(\frac{\partial q^\mu}{\partial \tilde{q}^\nu} \tilde{u}^\nu \right) = (1/a, 1/a^2, 0, 0) \quad (83)$$

and thus for the energy density

$$\langle T^{00} \rangle = \langle \epsilon(t, \chi) \rangle (u^0)^2 = \frac{a_0^2 \tilde{\omega}^2}{2a^4(t) S_K^2(\chi)}. \quad (84)$$

Because of $T^{0\mu} T^{0\nu} g_{\mu\nu} = 0$ this is also the magnitude of the energy-flux density (or Poynting vector). So the total radiation power at the source of the emission at χ_s is

$$P_e = \frac{a_0^2 \tilde{\omega}^2}{2a^4(t_e) S_K^2(\chi_s)} 4\pi a^2(t_e) S_K^2(\chi_s) = \frac{4\pi a_0^2 \tilde{\omega}^2}{2a^2(t_e)}. \quad (85)$$

Here $4\pi a^2(t_e) S_K^2(\chi_s)$ is the surface of the sphere $t = t_e = \text{const}$, $\chi = \chi_s = \text{const}$ at the source. This implies $a_0^2 \tilde{\omega}^2 = 2a^2(t_e) P_e / (4\pi)$ and thus the energy-flux density at the place of observation at $\chi = \chi_o$ is

$$L = \langle T^{00} \rangle_{t=t_o, \chi} = \frac{P_e a^2(t_e)}{4\pi a^4(t_o) S_K^2(\chi_o)}. \quad (86)$$

With (59) we can express $a(t_e)$ with the red-shift parameter z ,

$$L = \frac{P_e}{4\pi(1+z)^2 a^2(t_o) S_K^2(\chi)}. \quad (87)$$

The **luminosity distance** d_L is then defined by

$$L = \frac{P_e}{4\pi d_L^2}, \quad (88)$$

i.e., as the distance of an observer from a spherically symmetric light source with radiation power P_0 in flat Minkowski space resulting in the observed intensity (87), which leads to

$$d_L = (1+z) a(t_o) S_K(\chi). \quad (89)$$

To express $a(t_o)$ in terms of the redshift z , one needs a specific cosmological model defined by a postulated “**matter content**” of the universe and the solution of the corresponding Friedmann equations.

6 Electrodynamics in Schwarzschild spacetime

Another important application of the electrodynamics in general relativity are the historically important empirical tests of general relativity in our solar system. Here one takes the **Schwarzschild metric**

due to the presence of the Sun to define the spacetime, which reads in terms of the Schwarzschild coordinates $(t, r, \vartheta, \varphi)$

$$ds^2 = (1 - 2m/r)c^2 dt^2 - (1 - 2m/r)^{-1} dr^2 - r^2(d\vartheta^2 + \sin^2 \vartheta d\varphi^2). \quad (90)$$

Here $m = GM_{\text{Sun}}/c^2 = r_{\text{S}}/2$. Here we use this spacetime for $r > r_{\text{S}}$ outside of the Sun, for which $r_{\text{S}} \simeq 3$ km. The Schwarzschild coordinates have a coordinate singularity at $r = r_{\text{S}}$, which defines an **event horizon**. In the following we shall consider the gravitational **red shift** of spectral lines, the **deflection of light** in the gravitational field of the Sun (which was the first confirmation of a generic prediction of GR by Einstein from the solstice expeditions in 1919), and **Shapiro delay**.

The Lagrangian determining the lightlike geodesics reads

$$L = \frac{1}{2} \left[\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\vartheta}^2 - r^2 \sin^2 \vartheta \dot{\varphi}^2 \right]. \quad (91)$$

6.1 Gravitational redshift

For the demonstration of the gravitational redshift, i.e., of the shift of the spectral lines of light emitted from the Sun, it is sufficient to note that

$$p_t = k_0 = \frac{\partial L}{\partial \dot{t}} = c^2 \left(1 - \frac{2m}{r}\right) \dot{t} = \text{const}, \quad (92)$$

because L does not explicitly depend on t , i.e., $\partial_t L = 0$.

For an observer at rest on Earth, $r_o = 1 \text{ AU} = \text{const}$,

$$u^\mu = \left(1 - \frac{2m}{r}\right)^{-1/2} (1/c, 0, 0, 0). \quad (93)$$

This implies that

$$\omega = k_\mu u^\mu = \frac{1}{c} k_0 \left(1 - \frac{2m}{r}\right)^{-1/2}. \quad (94)$$

This implies that for $r = r_e$, where r_e denotes the radial coordinate localizing the photosphere of the Sun

$$\omega_e = \frac{1}{c} k_0 \left(1 - \frac{2m}{r_e}\right)^{-1/2} \quad (95)$$

and for the observer on Earth

$$\omega_o = \frac{1}{c} k_0 \left(1 - \frac{2m}{r_o}\right)^{-1/2} \quad (96)$$

which implies that

$$\frac{\omega_o}{\omega_e} = \sqrt{\frac{1 - 2m/r_e}{1 - 2m/r_o}}. \quad (97)$$

Thus one expects a **red shift** (since $r_e < r_o$). Unfortunately this red shift of spectral lines of the sunlight could not be observed yet, although already the famous Einstein tower at Caputh close to Potsdam was specifically erected to measure this prediction of GR.

The gravitational redshift could, however, be observed on Earth using a laser and Mössbauer spectroscopy for the first time by Pound and Rebka in 1959. Here, of course $m = m_E GM_E/c^2 = g r_E^2/c^2$ (with $g \simeq 9.81\text{m/s}^2$), and we can write $r_e = r_E + h$ expand (97) to linear order in h :

$$\frac{\omega_o}{\omega_e} = 1 - z = \sqrt{\frac{1 - 2g r_E/c^2}{1 - 2g r_E^2/(r_E + h)}} \simeq 1 - \frac{gh}{c^2}. \quad (98)$$

This prediction has been confirmed by Pound and Rebka with an accuracy of about 10% being refined to 1% in 1964.

6.2 Deflection of light on the Sun

One of the first tests of GR, leading to Einstein become a celebrity in 1919, was the confirmation of his prediction of the deflection of light due to the gravitational field of the Sun. Here we have find the null geodesic for a light ray moving non-centrally. We use the Lagrangian (91). Due to spherical symmetry it is clear that the motion will be in a plane due to conservation of the corresponding “angular momentum”. This implies that we can set $\vartheta = 0$ by choosing the angular coordinates according to the initial conditions in this way. To see that this is indeed compatible with the equations of motion, we evaluate the Euler-Lagrange equation for ϑ :

$$p_\vartheta = \frac{\partial L}{\partial \dot{\vartheta}} = -r^2 \dot{\vartheta}, \quad \dot{p}_\vartheta = \frac{\partial L}{\partial \vartheta} = -r^2 \sin \vartheta \cos \vartheta \dot{\varphi}^2. \quad (99)$$

Indeed this is compatible with $\vartheta = \pi/2 = \text{const.}$

This reduces the Lagrangian to

$$L = \frac{1}{2} \left[\left(1 - \frac{2m}{r}\right) c^2 \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2 \dot{\varphi}^2 \right]. \quad (100)$$

Since L is independent of t and φ the corresponding canonical momenta,

$$p_t = \frac{\partial L}{\partial \dot{t}} = \left(1 - \frac{2m}{r}\right) c^2 \dot{t} = \text{const}, \quad (101)$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = -r^2 \dot{\varphi} = -h = \text{const}. \quad (102)$$

Finally we use that $H = L = \text{const} = 0$, because we look for null geodesics. Using (101) and (102) in (100) then yields

$$\frac{p_t^2}{c^2} \left(1 - \frac{2m}{r}\right)^{-1} - \left(1 - \frac{2m}{r}\right) \dot{r}^2 - \frac{h^2}{r^2} = 0. \quad (103)$$

Multiplying with $(1 - 2m/r)$ and introducing φ as the independent variable instead of λ , as well $u = 1/r$ gives

$$\frac{p_t^2}{c^2} - h^2 u'^2 - h^2 u^2 (1 - 2mu) = 0, \quad (104)$$

where $u' = du/d\varphi$. Taking the derivative of this equation with respect to φ and using that $\dot{u} \neq 0$ for our problem, we find

$$u'' + u = 3mu^2. \quad (105)$$

Let $D = r_{\min}$ the “perihelion distance” of the light ray from the Sun and using that $\epsilon = 3m/r_{\min} = 3mu_{\max} \ll 1$ for a light ray just gracing the rim of the Sun shows that the right-hand side of Eq. (105) is a small perturbation, we expand the solution in powers of ϵ ,

$$u = u_0 + \epsilon u_1 + \mathcal{O}(\epsilon^2). \quad (106)$$

Plugging this into (105) we get in leading order of ϵ

$$u_0'' + u_0 = 0 \Rightarrow u(\varphi) = u_{\max} \sin \varphi. \quad (107)$$

This is a straight line as to be expected from the limit $m = 0$, where the Schwarzschild metric just becomes the Minkowski metric with no gravitational field present.

For the first-order correction we find

$$u_1'' + u_1 = \frac{\epsilon}{u_{\max}} u_0^2 = u_{\max} \sin^2 \varphi = \frac{1}{2} u_{\max} [1 - \cos(2\varphi)]. \quad (108)$$

The general solution of this inhomogeneous linear differential equation is

$$u_1(\varphi) = \frac{1}{2} u_{\max} + \frac{1}{6} \cos(2\varphi) + A \cos \varphi + B \sin \varphi. \quad (109)$$

The constants of integration, A and B , are determined by making $u_1(\pi/2) = 0$ and $\dot{u}_1(\pi/2) = 0$, so that at $\varphi = \pi/2$ we have $u(\pi/2) = u_0(\pi/2) = u_{\max}$. This finally leads to

$$u(\varphi) = u_{\max} \left(1 - \frac{\epsilon}{3}\right) \sin \varphi + \frac{\epsilon}{2} u_{\max} \left[1 + \frac{1}{3} \cos(2\varphi)\right]. \quad (110)$$

For $r \rightarrow \infty$, i.e., $u \rightarrow 0$, the angle $\varphi = \delta \ll 1$ and thus from (110) with $\sin \delta \simeq \delta = \mathcal{O}(\epsilon)$ and $\cos(2\delta) \simeq 1 + \mathcal{O}(\epsilon^2)$. So we get

$$\delta = -2\epsilon/3 = -2mu_{\max}. \quad (111)$$

The total deflection thus is

$$2\delta = -4mu_{\max} = -\frac{4m}{r_{\min}} \simeq 1.75'', \quad (112)$$

where we used $m = r_S/2 = 1.5$ km and $r_{\min} = R_{\text{Sun}} = 6.963 \cdot 10^6$ km.

A The energy-momentum tensor of the free electromagnetic field

Luminosity distance in the FLRW spacetime

FLRW Metric

```

In[1]:= q = {tau, chi, th, ph};
In[2]:= g = atil[tau]^2
        {{1, 0, 0, 0}, {0, -1, 0, 0}, {0, 0, -S[chi]^2, 0}, {0, 0, 0, -S[chi]^2 Sin[th]^2}};
In[3]:= MatrixForm[g]
Out[3]/MatrixForm=

$$\begin{pmatrix} \text{atil}[\tau]^2 & 0 & 0 & 0 \\ 0 & -\text{atil}[\tau]^2 & 0 & 0 \\ 0 & 0 & -\text{atil}[\tau]^2 S[\text{chi}]^2 & 0 \\ 0 & 0 & 0 & -\text{atil}[\tau]^2 S[\text{chi}]^2 \text{Sin}[\text{th}]^2 \end{pmatrix}$$

In[4]:= dq = {dtau, dchi, dth, dph};
In[5]:= FullSimplify[dq.g.dq]
Out[5]= -atil[tau]^2 (dchi^2 - dtau^2 + S[chi]^2 (dth^2 + dph^2 Sin[th]^2))

Lightlike unit vector in chi direction
In[6]:= ucov = {1, -1, 0, 0}
Out[6]= {1, -1, 0, 0}
In[7]:= gcontra = Inverse[g];
In[8]:= u = gcontra.ucov
Out[8]=  $\left\{ \frac{1}{\text{atil}[\tau]^2}, \frac{1}{\text{atil}[\tau]^2}, 0, 0 \right\}$ 
In[9]:= u.g.u
Out[9]= 0

Christoffel Symbols
In[10]:= christ =
Table[Table[Sum[1/2 gcontra[[i,i]][[m,i]] (D[g[[m,i]][[k,i]], q[[l,i]] + D[g[[m,i]][[l,i]], q[[k,i]] -
D[g[[k,i]][[l,i]], q[[m,i]])), {m, 1, 4}], {k, 1, 4}], {l, 1, 4}], {i, 1, 4}];

```

Energy-momentum tensor of the em. field

```

In[11]:= Acov = {0, 0, 0, a0 Cos[omtil (tau - chi)]}
Out[11]= {0, 0, 0, a0 Cos[omtil (-chi + tau)]}

In[12]:= Fcov = Table[D[Acov[[nu]], q[[mu]]] - D[Acov[[mu]], q[[nu]]], {mu, 1, 4}, {nu, 1, 4}
Out[12]= {{0, 0, 0, -a0 omtil Sin[omtil (-chi + tau)]}, {0, 0, 0, a0 omtil Sin[omtil (-chi + tau)]},
          {0, 0, 0, 0}, {a0 omtil Sin[omtil (-chi + tau)], -a0 omtil Sin[omtil (-chi + tau)], 0, 0}}

In[13]:= Fcontra = gcontra . Fcov . gcontra
Out[13]= {{0, 0, 0,  $\frac{a0 \text{omtil Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}}{\text{atil[tau]}^4 \text{S[chi]}^2}$ },
          {0, 0, 0,  $\frac{a0 \text{omtil Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}}{\text{atil[tau]}^4 \text{S[chi]}^2}$ },
          {0, 0, 0, 0}, {- $\frac{a0 \text{omtil Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}}{\text{atil[tau]}^4 \text{S[chi]}^2}$ ,
          - $\frac{a0 \text{omtil Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}}{\text{atil[tau]}^4 \text{S[chi]}^2}$ , 0, 0}}

In[14]:= Sum[Fcontra[[a]][[be]] * Fcov[[a]][[be]], {a, 1, 4}, {be, 1, 4}]
Out[14]= 0

In[15]:= Tcontra = FullSimplify[Fcontra . g . Fcontra +
          1 / 4 Sum[Fcontra[[a]][[be]] * Fcov[[a]][[be]], {a, 1, 4}, {be, 1, 4}] gcontra]
Out[15]= {{ $\frac{a0^2 \text{omtil}^2 \text{Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}^2}{\text{atil[tau]}^6 \text{S[chi]}^2}$ ,  $\frac{a0^2 \text{omtil}^2 \text{Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}^2}{\text{atil[tau]}^6 \text{S[chi]}^2}$ ,
          0, 0}, { $\frac{a0^2 \text{omtil}^2 \text{Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}^2}{\text{atil[tau]}^6 \text{S[chi]}^2}$ ,
           $\frac{a0^2 \text{omtil}^2 \text{Csc[th]}^2 \text{Sin[omtil (-chi + tau)]}^2}{\text{atil[tau]}^6 \text{S[chi]}^2}$ , 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[16]:= Table[u[[mu]] * u[[nu]], {mu, 1, 4}, {nu, 1, 4}]
Out[16]= {{ $\frac{1}{\text{atil[tau]}^4}$ ,  $\frac{1}{\text{atil[tau]}^4}$ , 0, 0}, { $\frac{1}{\text{atil[tau]}^4}$ ,  $\frac{1}{\text{atil[tau]}^4}$ , 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

Trace of the energy-momentum tensor:

In[17]:= Sum[(Tcontra . g)[[mu]][[mu]], {mu, 1, 4}]
Out[17]= 0

Local energy conservation

```



```

In[18]:= Table[Sum[D[Tcontra[[a1]][[ga]], q[[a1]]], {a1, 1, 4}] +
  Sum[christ[[a1]][[a1]][[mu]] × Tcontra[[mu]][[ga]] + christ[[ga]][[a1]][[mu]] × Tcontra[[a1]][[mu]],
  {a1, 1, 4}, {mu, 1, 4}], {ga, 1, 4}]
Out[18]= {0, 0, 0, 0}

```

Literatur

- [ABS75] R. Adler, M. Bazin, M. Schiffer, *Introduction to general relativity*, 2. Aufl., McGraw-Hill Inc., New York (1975).
- [CK74] J. M. Cohen, L. S. Kegeles, *Electromagnetic fields in curved spaces - a constructive procedure*, Phys. Rev. D **10**, 1070 (1974).
<https://doi.org/10.1103/PhysRevD.10.1070>
- [Fli12] T. Fließbach, *Allgemeine Relativitätstheorie*, 6. Aufl., Springer-Verlag, Berlin, Heidelberg (2012).
- [LL92] L. D. Landau, E. M. Lifschitz, *Klassische Feldtheorie*, Bd. 2 von *Lehrbuch der Theoretischen Physik*, Akademie Verlag, Berlin (1992).
- [LL96] L. D. Landau, E. M. Lifshitz, *Course of Theoretical Physics 2, The Classical Theory of Fields*, 4. Aufl., Butterworth Heinemann, Oxford (1996).
- [Som54] A. Sommerfeld, *Optics (Lectures on Theoretical Physics, vol. IV)*, Academic Press, New York (1954).
- [Som78] A. Sommerfeld, *Vorlesungen über Theoretische Physik IV, Optik*, Verlag Harri Deutsch (1978).