Exercise Sheet #9

Oren Neumann <neumann@itp.uni-frankfurt.de>
Daniel Nevermann <nevermann@itp.uni-frankfurt.de>

There will be no homework sheet over the holidays period, to allow you to start working on your projects.

Problem 1 (*Spatio-temporal Turing instability, 10pt*)

Consider a generic reaction-diffusion system with two interacting components:

$$\dot{\rho} = f(\rho, \sigma) + D_\rho \Delta \rho$$
$$\dot{\sigma} = g(\rho, \sigma) + D_\sigma \Delta \sigma \,,$$

where the densities $\rho(x,t), \sigma(x,t)$ define the state of the system, f, g are the reaction terms and $D_\rho, D_\sigma > 0$ denote the diffusion constants. We assume that the reaction terms have a stable focus $f(\rho_0, \sigma_0) = g(\rho_0, \sigma_0) = 0$ with steady state densities ρ_0, σ_0.

Assume that $f_\rho < 0$ and $g_\sigma > 0$, where e. g. $f_\rho = \partial f / \partial \rho|_{(\rho_0, \sigma_0)}$ denotes the derivative evaluated at the fixpoint.

Using a Fourier expansion of pertubations around the fixpoint (cf. Eq. (4.30) in the lecture notes), show that a Turing instability can only occur, if the following condition holds:

$$2|f_\sigma g_\rho| > |f_\rho g_\sigma| \,.$$

That means that the determinant of the linearised system can only be negative, if a critical ratio of the diffusion constants $D_\rho / D_\sigma > 0$ exists.

Problem 2 (*FitzHugh–Nagumo system with diffusion, 10pt*)

The FitzHugh–Nagumo system is a simplified model of a spiking neuron describing the generation of a voltage spike by a non-linear conductance term and a linear recovery variable. Including a diffusive term the dimensionless FitzHugh–Nagumo equations read:

$$\dot{u} = D\Delta u - u(u+\alpha)(u-1) - v \qquad (1)$$
$$\dot{v} = \Delta v + u - v \,, \qquad (2)$$

where the parameter $-3 < \alpha < 1$ and the relative diffusion constant $D < 1$. Here we are going to study the onset of the Turing instability in the FitzHugh-Nagumo reaction–diffusion system.

a) Investigate the reaction system by neglecting the diffusion terms ($\Delta u = \Delta v = 0$). Find the fixed point(s) of the reaction system and analyse the stability.

b) Now add the diffusive terms ($\Delta u \neq 0$, $\Delta v \neq 0$) and investigate the stability of the fixed point(s) you found before. Therefore use a plain wave ansatz with wave numbers $k > 0$.

c) Determine the critical value of α as a function of the diffusion constant D for which the first wave vector becomes unstable. Also calculate the first wave vector to become unstable.

d) Make a phase space sketch for the FitzHugh-Nagumo system without diffusion. Include the fixed point(s), the two nullclines ($\dot{u} = 0$, $\dot{v} = 0$) and draw some particular trajectories. How does this help to understand pattern formation in the system?