Exercise Sheet #7

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Problem 1 (*Langevin Equation, 10 points*)

Consider the Langevin equation:

$$m\dot{v} = -\gamma m v + \eta(t) \tag{1}$$

where the $\eta(t)$ term denotes white noise:

$$\langle \eta(t)\eta(t')\rangle = Q\delta(t-t') \tag{2}$$

$$\langle \eta(t)\rangle = 0\ . \tag{3}$$

Show that, in thermal equilibrium, the following holds:

$$\langle v(t)\eta(t)\rangle = 2k_{\mathrm{B}}\gamma T\ , \tag{4}$$

using the equipartition theorem: $\frac{1}{2}m\langle v^2(t)\rangle = \frac{k_{\mathrm{B}}T}{2}$.

Problem 2 (*First Passage Time and Diffusion, 10 points*)

Consider a particle with Brownian motion moving in one dimension on a horizontal bar that stretches between $x=-a$ and $x=a$. If the particle exceeds these boundaries, it falls off the bar. We would like to know the probability distribution for the time this happens. In other words, we want to find the "first passage time" for passing either of the two boundaries $-a$ and a. The probability distribution for the particle is described by the diffusion equation

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2}\ . \tag{5}$$

Since the two boundaries are "absorbing", we enforce $p(|x|\geq a, t) = 0$ for all t. The particle starts at $x(0)=0$, that is, $p(x,0)=\delta(x)$.

a) Show that the solution for $p(x,t)$ is

$$p(x,t) = \sum_{n=1}^{\infty}\frac{1}{a}\sin\frac{n\pi}{2}e^{-n^2\pi^2 Dt/(4a^2)}\sin\left[\frac{n\pi(x+a)}{2a}\right] \tag{6}$$

by using the fundamental solution of the diffusion equation and imposing the constraints $p(a,t)=0$ and $p(-a,t)=0$.

b) If $\int_{-a}^{a}p(x,t)\mathrm{dx}$ is the probability distribution of being *somewhere* within the boundaries, derive an expression for the probability of falling out of these boundaries within a small time interval $[t, t+\Delta t]$. If this probability is written as $f(t)\Delta t$, then $f(t)$ is the first-passage time density that we are looking for.

c) Calculate the average time the particle stays within the boundaries as a function of D and a.

d) (Optional) Try the same procedure with the case where the allowed region is $x\in[-\infty, a)$ and the particle starts at zero. That is, analyze the case where the particle can only fall over one edge at $x=a$, coming from the left.