Exercise Sheet #6

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Problem 1 (*Hausdorff Dimension, 5pt*)

Calculate the Hausdorff dimension of a straight line and of the Cantor set.

The Cantor ternary set is created by repeatedly deleting the open middle thirds of a set of line segments. First, one removes the open middle third (1/3, 2/3) from the interval [0, 1], leaving two line segments: [0, 1/3] and [2/3, 1]. Then, the open middle third of each of these remaining segments is also removed, leaving four line segments: [0, 1/9], [2/9, 1/3], [2/3, 7/9] and [8/9, 1]. Then the remaining four segments get their open middle third removed and the process is repeated infinitely.

Problem 2 (*Lorenz system, 15pt*)

a) Analyse the fixpoints of the Lorenz system given by

$$\begin{aligned}\dot{x} &= s(y-x)\\ \dot{y} &= x(r-z)-y\\ \dot{z} &= xy-bz\end{aligned}$$

and their stability as a function of the Rayleigh number $r > 0$ with parameters $s = 10$ and $b = 8/3$ fixed. Make a sketch of the $x(r)$ bifurcation diagram. (*Hint*: You may calculate the the eigenvalues numerically.)

b) Show analytically that all fixpoints are unstable for $r > r_{\mathrm{H}} = s(s+b+3)/(s-b-1)$. (*Hint*: You do not necessarily need a full solution for the eigenvalue spectra, since you are only looking for this particular transition point.)

c) What kind of bifurcation does the system undergo at $r = r_{\mathrm{H}}$ and what kind of dynamics are present in the system for $r > r_{\mathrm{H}}$?

d) Evaluate numerically the Pointcare map of the Lorenz model with $\sigma = 10$ and $b = 8/3$, for:

  i) $r = 22$ (regular regime) and the plane z = 21,
  ii) $r = 28$ (chaotic regime) and the plane z = 27.

Make a plot of the maps and mark the fixpoints. What is the behavior of the points in each map as they evolve? (particularly in relation to the fixpoints).