# Exercise Sheet #10

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**Problem 1** (*Car-following Models, 10pts*)

Consider the Bando and Hasebe velocity difference car following model:

$$\ddot{x}_n(t) = a\left[V\left(x_{n+1}(t) - x_n(t)\right) - \dot{x}_n(t)\right]$$

where $x_n(t)$ is the position of the n-th car in the row ($n = 1, 2, ..., N$). $N$ is the total number of vehicles and $a$ is a constant representing the drivers' sensitivity (which is independent of $n$). The legal velocity function $V\left(x_{n+1} - x_n\right)$ of vehicle number $n$ depends on the following distance of the preceding vehicle. $V$ is a monotonically increasing function with an upper bound.

Study the stability of the steady state solution when each vehicle has the same constant speed and the same following distance. Hint: use periodic boundary conditions and linearize the dynamics around the steady state solution.

**Problem 2** (*2D Ising Model, 10pts*)

The two-dimensional Ising model exhibits a phase transition with respect to its temperature. The model is defined by a 2D square lattice of spins $\sigma = \pm 1$ with the Hamiltonian:

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \cdot \sigma_j$$

where $\langle i, j \rangle$ are all nearest neighbour pairs, with periodic boundary conditions.

We will use the Metropolis algorithm to calculate the phase transition temperature numerically. This algorithm samples states by performing a biased random walk, jumping from one state to the next according to the ratio of probabilities for each state. The algorithm is implemented as follows:

- starting from an initial state, choose a new candidate state by flipping the spin of one random site.
- Calculate the ratio $\alpha$ of the new and old state probabilities. Remember that the probability of a state $i$ is $\frac{1}{Z}\exp(-\beta E_i)$, so the ratio is $\alpha = \exp(-\beta \cdot (E_{new} - E_{old}))$.
- Use this ratio to determine whether to jump to the new candidate state, or remain in the old state. Picking a uniformly random number in $[0, 1]$, check if this number is smaller than $\alpha$. If it is, change to the new state. Otherwise, keep the system unchanged.
- Repeat this for many iterations. You can calculate averages of observables by sampling them every $N$ iterations (= one sweep), $N$ being the number of sites.

Write your version of this algorithm and calculate the average energy $\langle E \rangle$ and specific heat $\langle C_v \rangle = \frac{\beta^2}{N}(\langle E^2 \rangle - \langle E \rangle^2)$. Do this for different system lengths $L = 8, 16, 32$ and plot these averages against the temperature $T = 0.1, 0.2, 0.3 ... 3.9, 4$. Run the algorithm for at least $10,000$ sweeps ($= N$ iterations), making sure to discard the first $1,000$ sweeps to let the system thermalize. You can set $\beta \cdot J = 1$. What is the approximate temperature of the phase transition, seen in the specific heat curve?