Problem 1  

(Booleans Networks with Synchronous Update)  

(a) Consider a Kauffman network with $N = 3$ sites, indegree $K = 1$, and a cyclic connectivity, i.e. $\sigma_1 = f_1(\sigma_2)$, $\sigma_2 = f_2(\sigma_3)$, $\sigma_3 = f_3(\sigma_1)$. Make a plot of the network and find for the following cases of couplings all the cycles and their basin of attraction:

(i) $f_1 = f_2 = f_3 = \text{identity}$,
(ii) $f_1 = f_2 = f_3 = \text{negation}$,
(iii) $f_1 = f_2 = \text{negation}$, $f_3 = \text{identity}$.

(b) Now consider the following boolean network with $N = 4$ sites:

![Diagram](image)

Assuming that all the coupling functions are generalised XOR functions, i.e. the function returns 1 (0), if the number of '1s' in the input is odd (even), find all cycles of the network.

Problem 2  

(Mutual Information and Conditional Entropy)  

The mutual information of two random variables $X$ and $Y$ can be defined as:

$$I(X,Y) = H(X) + H(Y) - H(X,Y)$$

(a) Using the property $H(X|Y) = -\int p(x,y) \log(p(x|y)) \, dx \, dy$, show that $I(X,Y)$ can also be defined in terms of the conditional entropy:

$$I(X,Y) = H(X) - H(X|Y)$$

(b) Calculate the conditional entropy $H(X|Y)$ for two completely independent random variables $X$ and $Y$.

How does the mutual information look in such a case?
Problem 3  (Noisy communication)  6 Pts

Consider a communication channel with $X = (x_1, \ldots, x_n)$, $x_i \in \{a,b,c,d\}$ being the input channel and $Y = (y_1, \ldots, y_n)$ the output channel. The communication is found to be noisy:

$$x_i = a \quad \Rightarrow \quad y_i = a$$

$$x_i \neq a \quad \Rightarrow \quad y_i = \begin{cases} \text{a with } p \\ \text{x with } 1 - p \end{cases}$$

(a) Calculate the entropies $H(X)$, $H(Y)$ of the single channels, as well as the joint entropy $H(X,Y)$.

(b) Compare the conditional entropies $H(Y|X)$ and $H(X|Y)$ to the mutual information $I(X,Y)$ of the noisy communication.

(c) Calculate all of the previous properties in units of bits, so use the base $b = 2$ for the logarithm, for the case $p = 1/4$.