Exercise Sheet #13

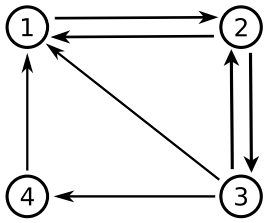
Hendrik Wernecke <wernecke@th.physik.uni-frankfurt.de>
Philip Trapp <trapp@th.physik.uni-frankfurt.de>

Problem 1 (*Boolean Networks with Synchronous Update*) 10 Pts

(a) Consider a Kauffman network with $N = 3$ sites, indegree $K = 1$, and a cyclic connectivity, i. e. $\sigma_1 = f_1(\sigma_2)$, $\sigma_2 = f_2(\sigma_3)$, $\sigma_3 = f_3(\sigma_1)$.
Make a plot of the network and find for the following cases of couplings all the cycles and their basin of attraction:

(i) $f_1 = f_2 = f_3 =$ identity,
(ii) $f_1 = f_2 = f_3 =$ negation,
(iii) $f_1 = f_2 =$ negation, $f_3 =$ identity.

(b) Now consider the following boolean network with $N = 4$ sites:



Assuming that all the coupling functions are generalised XOR functions, i. e. the function returns 1 (0), if the number of '1s' in the input is odd (even), find all cycles of the network.

Problem 2 (*Mutual Information and Conditional Entropy*) 4 Pts

The mutual information of two random variables X and Y can be defined as:

$$I(X, Y) = H(X) + H(Y) - H(X, Y)$$

(a) Using the property $H(X|Y) = -\int p(x, y) \log(p(x|y)) \mathrm{d}x\mathrm{d}y$, show that $I(X, Y)$ can also be defined in terms of the conditional entropy:

$$I(X, Y) = H(X) - H(X|Y)$$

(b) Calculate the conditional entropy $H(X|Y)$ for two completely independent random variables X and Y.
How does the mutual information look in such a case?

Problem 3 (*Noisy communication*) 6 Pts

Consider a communication channel with $X = (x_1, \ldots, x_n)$, $x_i \in \{\text{a,b,c,d}\}$ being the input channel and $Y = (y_1, \ldots, y_n)$ the output channel. The communication is found to be noisy:

$$x_i = \text{a} \qquad \Rightarrow \qquad y_i = \text{a}$$

$$x_i \neq \text{a} \qquad \Rightarrow \qquad y_i = \begin{cases} \text{a} & \text{with } p \\ x_i & \text{with } 1-p \end{cases}$$

(a) Calculate the entropies $H(X)$, $H(Y)$ of the single channels, as well as the joint entropy $H(X,Y)$.

(b) Compare the conditional entropies $H(Y|X)$ and $H(X|Y)$ to the mutual information $I(X,Y)$ of the noisy communication.

(c) Calculate all of the previous properties in units of bits, so use the base $b = 2$ for the logarithm, for the case $p = 1/4$.