Exercise Sheet #10

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Problem 1 (Random Walks) 5 Pts

A drunken man is trying to walk along a straight sidewalk. He has already lost the notion of direction, so that he has an equal probability to make a step forward as back. Determine the displacement distribution after $t$ steps, $P(x,t)$, its average displacement $\langle x \rangle$ (how far is he away from his starting point), and the variance $\langle x^2 \rangle - \langle x \rangle^2$ (the square of how much he actually walked) of its path. Find the exact result for $P(x,t)$ for a small number of steps $t \sim 10$. Find and an approximation for long walking time $t \to \infty$.

Problem 2 (Fokker-Planck equation) 5 Pts

Consider a random walk in one dimension, where instead of equally probable steps in both directions, you have:

- probability of a step to the left $q$
- probability of a step to the right $1 - q$

(a) Write down the master equation for the probability $P(x,t)$ to be at position $x$ at time $t$.

(b) Suppose now, that steps $dx \to 0$ are infinitesimal and we are in the continuous time limit $dt \to 0$. Derive a differential equation for the time evolution of the probability $P(x,t)$.

*Hint:* You should get the Fokker-Planck equation in the general form.

(c) Find the corresponding expression for an equilibrated (standard) random walk.

Problem 3 (Solution of Fokker-Planck equation in a harmonic potential with additional constant drift) 5 Pts

The Fokker-Planck equation describes the evolution of a probability density function $P(x,t)$ under diffusion and ballistic transport

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} (v_d P(x,t)) + \frac{\partial^2}{\partial x^2} (DP(x,t)),$$

with a drift velocity $v_d$ and the diffusion coefficient $D$. Assume that
(1) the density distribution is stationary $\partial P(x,t)/\partial t = 0$,

(2) the drift velocity $v_d = v_o + \frac{F(x)}{\gamma m}$ is given by a force $F(x) = -\frac{dV(x)}{dx}$ in a potential $V$ (with $m$, $\gamma$ denoting here the mass and the friction coefficient) and an additional constant velocity $v_o$,

(3) the potential $V(x) = fx^2/2$ is harmonic.

Calculate the solution of the Fokker-Planck equation.

**Problem 4 (Pareto distribution)**

The Pareto Type I distribution defined by

$$p(x) = \begin{cases} \alpha x_m^\alpha x^{-\alpha-1} & \text{for } x \geq x_m \\ 0 & \text{for } x < x_m \end{cases}$$

where $x_m > 0$ is the minimum possible value of for a random variable $x$, and $\alpha$ is a positive parameter. When this distribution is used to model the distribution of wealth, then the parameter $\alpha$ is called the Pareto index.

(a) Calculate the "raw" moments $\mu_n = \int_R dx p(x) x^n$ of the distribution.

(b) Calculate the median of the Pareto distribution and compare it to the mean.

(c) Calculate the cumulative probability distribution corresponding to the Pareto distribution.