Exercise Sheet #8

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Problem 1 (*Distance scaling*) 8 Pts

The normal form of a limit cycle in polar coordinates (r, φ) is given by

$$\dot{r} = r(\Gamma^2 - r^2)$$
$$\dot{\varphi} = \Omega(r)$$

with the limit cycle at $r^* = \Gamma$ and an arbitrary smooth function $\Omega(r)$ for the angular velocity.

(a) Approximate the dynamics locally close to the attractor at $r = \Gamma + \epsilon$ and do a Taylor expansion. That means you should identify the coeficients a, b, c in

$$\dot{\varphi} = a + b\epsilon$$
$$\dot{\epsilon} = -c\epsilon\ .$$

(2 Pts)

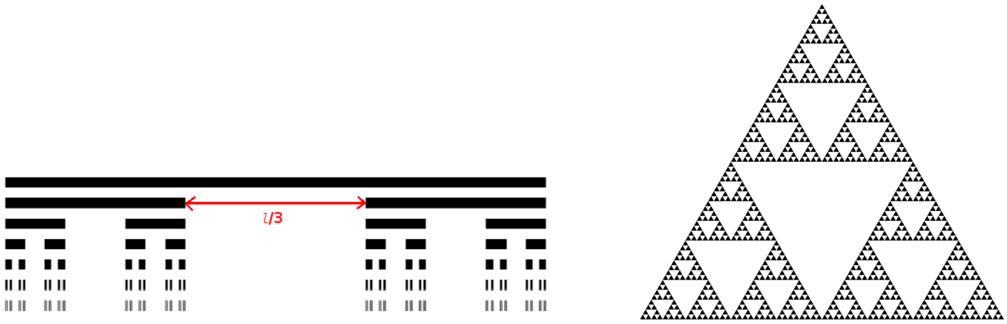
(b) For simplicity we now change the notation $(\varphi, \epsilon) \to (x, y)$. Solve the linearised system for arbitrary initial conditions (x_o, y_o). (2 Pts)

(c) Now calculate the Euclidean distance d of two trajectories that start at (x_o, y_o) and $(x_o + \delta_x, y_o + \delta_y)$ respectively. The resulting expression $d(t, \delta_x, \delta_y)$ will depend on both time and the initial spatial displacement (δ_x, δ_y). (2 Pts)

(d) In order to generalise the expression for the distance, we assume that the initial distance $\delta = \sqrt{\delta_x^2 + \delta_y^2}$ is fixed. Average the distance over all configurations that fulfill this condition, i. e. compute

$$\langle d \rangle = \oint_{\mathcal{C}} \mathrm{d}s\ d(t, \delta_x, \delta_y)\ ,$$

where $\mathcal{C} = \{(\delta_x, \delta_y)\,|\,\delta^2 = \delta_x^2 + \delta_y^2\}$ is a circle with radius δ. How does the averaged distance $\langle d \rangle$ depend on the initial distance δ? (2 Pts)



(a) Cantor set (b) Sierpinski triangle

Figure 1: Sketches of Cantor set and Sierpinski triangle

Problem 2 (*Fractal Dimension*) 6 Pts

(a) The so called Cantor set is constructed by iteratively removing the middle third of line segments (see figure 1(a)).
Identify the fractal dimension of the Cantor set! (2 Pts)

(b) The Sierpinski triangle is an equilateral triangle, which is constructed by iteratively "cutting out" an inverted triangle from the center of the original triangle with half the length of the original triangle (see figure 1(b)).
Determine the fractal dimension of the Sierpinski triangle! (2 Pts)

(c) How does the fractal dimension of the Sierpinski triangle change, when the triangle is not equilateral (=gleichseitig) any more, but an isosceles (gleichschenkliges) triangle. (2 Pts)

Problem 3 (*Phase space contraction*) 6 Pts

Recall the dynamical system with a bifurcation on an invariant cycle, discussed on sheet 5, ex.2:

$$\begin{aligned} \dot{x} &= -y(x-a) - x(x^2+y^2-1) \\ \dot{y} &= x(x-a) - y(x^2+y^2-1)\ , \end{aligned}$$

with the bifurcation parameter a.
In polar coordinates this system yields:

$$\begin{aligned} \dot{r} &= -r(r^2-1) \\ \dot{\varphi} &= r\cos\varphi - a \end{aligned}$$

(a) Calculate the phase space contraction in both coordinate systems and compare the two expressions. Are they generally the same? (3 Pts)

(b) How does the phase space contraction behave in the fixpoints for both coordinate systems? (2 Pts)

(c) How does the phase space contract on the invariant circle $r = 1$? Compare the solutions of both coordinate systems! (1 Pts)