Exercise Sheet #4
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Problem 1  (Preferential Attachment and Internal Growth) 7 Pts

In networks such as the internet and social communities new connections
are not only created when new nodes are added, but also between already
existing nodes – a phenomenon called internal growth of a network. To model
this, generalise the preferential attachment model in the following way:

(i) at each time step one new vertex and \( m \) new edges are added

(ii) with probability \( \rho \in [0, 1] \) one of the new edges connects the new vertex
    and an existing vertex \( i \), which is selected with probability
    \( \Pi_i \propto k_i + C \)

(iii) with probability \( 1 - \rho \) one of the new edges connects the existing vertices
    \( j \) and \( l \), which are selected with probability \( \pi_{jl} \propto \Pi_j \Pi_l \)

Show that the degree distribution \( p_k \) follows a power law,

\[
p_k \propto k^{-\gamma},
\]

with the exponent \( \gamma = 1 + \frac{1}{1-\rho^2} \). What happens in the limit \( \rho \to 1 \)?

Problem 2  (Dynamical system) 7 Pts

Consider the following two dimensional dynamical system in \( (x, y) \in \mathbb{R}^2 \):

\[
\begin{align*}
\dot{x} &= x (a - 2x - y) \\
\dot{y} &= a - x - 2y,
\end{align*}
\]

with the real parameter \( a \).

(a) Find all fixpoints of the system. (2 Pts)

(b) Linearise the system around the fixpoints in order to determine their
    stability and find the stable/unstable manifolds. (2 Pts)

(c) Sketch the flow of the system once for \( a > 0 \) and for \( a < 0 \). (2 Pts)

(d) What kind of bifurcation do you observe when \( a \) changes its sign? (1 Pts)
Problem 3 \textit{(Driven harmonic oscillator – revised)} \hspace{1cm} 6 \text{ Pts}

Investigate the dynamics of the damped driven harmonic oscillator

\[ \ddot{x}(t) = -\omega_0^2 x(t) - \alpha \dot{x}(t) + f_o \cos(\Omega t + \varphi_o), \]  

\text{(1)}

as a dynamical system, where $\omega_0$ is the natural frequency of the oscillator and $\alpha$ is the damping factor. The oscillator is driven by an external harmonic force with amplitude $f_o$, driving frequency $\Omega$ and initial phase $\varphi_o$.

Calculate the fixed point(s) of the system, examine their stability and make a sketch of the flow in the phase space for each of the following cases:

(i) weak damping without driving: $\alpha/2 < \omega_0, f_o = 0$ (1.5 Pts)

(ii) strong damping without driving: $\alpha/2 > \omega_0, f_o = 0$ (1.5 Pts)

(iii) negative damping, i.e. energy up-take, without driving: $\alpha < 0, f_o = 0$ (1.5 Pts)

(iv) very weak damping in presence of ext. force: $\alpha/2 \ll \omega_0, f_o \neq 0$ (1.5 Pts)

Compare these results to the ones obtained in the first problem session.